On continued fraction algorithms

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Chapter V: Wieb Bosma and Ionica Smeets — An algorithm for finding approximations with optimal Dirichlet quality (`http://arxiv.org/abs/1001.4455`)
Submitted.

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

Contents

I

Introduction

In this chapter we introduce the basic notation and terminology used throughout this thesis. We give some well-known classical results without proofs. The main references for this chapter are [**48**], [**26**], [**51**], [**10**] and [**18**]. In this introduction we also state the most important results of this thesis and outline the remaining chapters.

1. Regular continued fractions

Every $x \in \mathbb{R}\backslash\mathbb{Q}$ has a unique regular continued fraction expansion (RCF-expansion) of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_n + \ldots}}} = [\,a_0; a_1,\, a_2, \ldots,\, a_n, \ldots\,]\,, \tag{I.1}$$

where a_0 is the integer part of x and where a_n for $n > 0$ is a positive integer. These so-called partial quotients a_n are defined below.

Remark I.2. Without loss of generality we may assume $x \in [0, 1) \setminus \mathbb{Q}$ and write $x = [\,a_1,\, a_2, \ldots,\, a_n, \ldots\,]$, omitting a_0. We do so from now on, unless explicitly stated otherwise. $\diamond$

Remark I.3. If $x \in \mathbb{Q}$ there are two RCF-expansions of $x = \frac{p}{q}$, both finite. In this case, the shorter RCF-expansion of x is obtained from Euclid's algorithm to find the greatest common divisor of the integers p and q; see Section 3.1.2 of [**10**]. $\diamond$

Definition I.4. The regular continued fraction operator $T : [0, 1) \to [0, 1)$ is defined by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ if } x \neq 0 \text{ and } T(0) = 0.$$

$\diamond$

Here $\left\lfloor \frac{1}{x} \right\rfloor$ denotes the integer part of $\frac{1}{x}$.

To find the continued fraction of x we put

$$T_0 = x,\, T_1 = T(x),\, T_2 = T(T_1),\, \cdots, T_n = T(T_{n-1}),\, \cdots$$

and we define the partial quotients a_n of x by

$$a_n = \left\lfloor \frac{1}{T_{n-1}} \right\rfloor, \quad n \geq 1.$$

Clearly,

$$a_n = \begin{cases} 1 & \text{if } T_{n-1} \in \left(\frac{1}{2}, 1\right) \\ \\ k & \text{if } T_{n-1} \in \left(\frac{1}{k+1}, \frac{1}{k}\right], \, k \geq 2 \end{cases}$$

and we find that

$$x = \cfrac{1}{a_1 + T_1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T_2}} = \cdots = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + T_n}}}.$$

Definition I.5. The nth convergent p_n/q_n of x is found by finite truncation in (I.1) after level n, i.e.

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} = [\, a_1, \, a_2, \ldots, \, a_n] \quad \text{for } n \geq 1.$$

◇

We have the following recurrence relations for p_n and q_n

$$\begin{cases} p_{-1} = 1; \, p_0 = 0; & p_n = a_n p_{n-1} + p_{n-2}, \quad n \geq 1, \\ q_{-1} = 0; \, q_0 = 1; & q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1. \end{cases} \tag{I.6}$$

The regular continued fraction convergents $\frac{p_n}{q_n} \in \mathbb{Q}$ of x converge to $x \in \mathbb{R} \setminus \mathbb{Q}$ and the approximations get better in each step, i.e.

$$\left| x - \frac{p_n}{q_n} \right| < \left| x - \frac{p_{n-1}}{q_{n-1}} \right|;$$

see Sections 5 and 6 of [**26**]. Furthermore, it holds that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{{q_n}^2}. \tag{I.7}$$

2. Approximation results

In 1798 Legendre proved the following result [**34**].

Theorem I.8. *If $p, q \in \mathbb{Z}$, $q > 0$, and $\gcd(p, q) = 1$, then*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \quad \textit{implies that} \quad \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix}, \quad \textit{for some } n \geq 0.$$

Legendre's Theorem is one of the main reasons for studying continued fractions, because it tells us that good approximations of irrational numbers by rational numbers are given by continued fraction convergents.

Definition I.9. Let $x \in [0,1)\backslash\mathbb{Q}$ and $\frac{p_n}{q_n} = [a_1, \ldots, a_n]$ be the nth regular continued fraction convergent of x. The approximation coefficient $\Theta_n = \Theta_n(x)$ is defined by

$$\Theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right|, \text{ for } n \geq 0.$$

$\diamond$

We usually suppress the dependence of Θ_n on x in our notation. The approximation coefficient gives a numerical indication of the quality of the approximation. It follows from (I.7) that $\Theta_n \leq 1$. For the RCF-expansion we have the following classical theorems by Borel (1905) [**3**] and Hurwitz (1891) [**19**] about the quality of the approximations.

Theorem I.10. *For every irrational number x, and every $n \geq 1$*

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{5}}.$$

The constant $1/\sqrt{5}$ is best possible.

Borel's result, together with the earlier result by Legendre implies the following result by Hurwitz.

Theorem I.11. *For every irrational number x there exist infinitely many pairs of integers p and q, such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}} \frac{1}{q^2}.$$

The constant $1/\sqrt{5}$ is best possible.

Remark I.12. If we replace $\frac{1}{\sqrt{5}}$ by a smaller constant C, then there are infinitely many irrational numbers x for which

$$\left| x - \frac{p}{q} \right| \leq \frac{C}{q^2}$$

holds for only finitely many pairs of integers p and q. An example of such a number is the small golden number $g = \frac{\sqrt{5}-1}{2}$. $\diamond$

3. Dynamical systems

We write t_n and v_n for the "future" and "past" of $\frac{p_n}{q_n}$, respectively,

$$t_n = [a_{n+1}, a_{n+2}, \ldots] \quad \text{and} \quad v_n = [a_n, \ldots, a_1]. \tag{I.13}$$

Furthermore, $t_0 = x$ and $v_0 = 0$. Due to the recurrence relation for the q_n in (I.6) it is easy to show by induction that $v_n = \frac{q_{n-1}}{q_n}$.

The approximation coefficients may be written in terms of t_n and v_n

$$\Theta_n = \frac{t_n}{1 + t_n v_n}, \quad \text{and} \quad \Theta_{n-1} = \frac{v_n}{1 + t_n v_n}, \quad n \geq 1; \tag{I.14}$$

see Section 5.1.2 of [**10**].

In order to study the sequence $(\Theta_n)_{n\geq 1}$ we introduce the two-dimensional operator $\mathcal{T}$.

Definition I.15. Put $\Omega = ([0,1)\setminus\mathbb{Q})\times[0,1]$. The operator $\mathcal{T}:\,\Omega\to\Omega$ is defined by

$$\mathcal{T}(x,y) := \left(T(x),\, \frac{1}{\left\lfloor\frac{1}{x}\right\rfloor + y}\right).$$

◇

For $x\in[0,1)\setminus\mathbb{Q}$, one has $\mathcal{T}^n(x,0) = (t_n, v_n), \quad n\geq 0$.

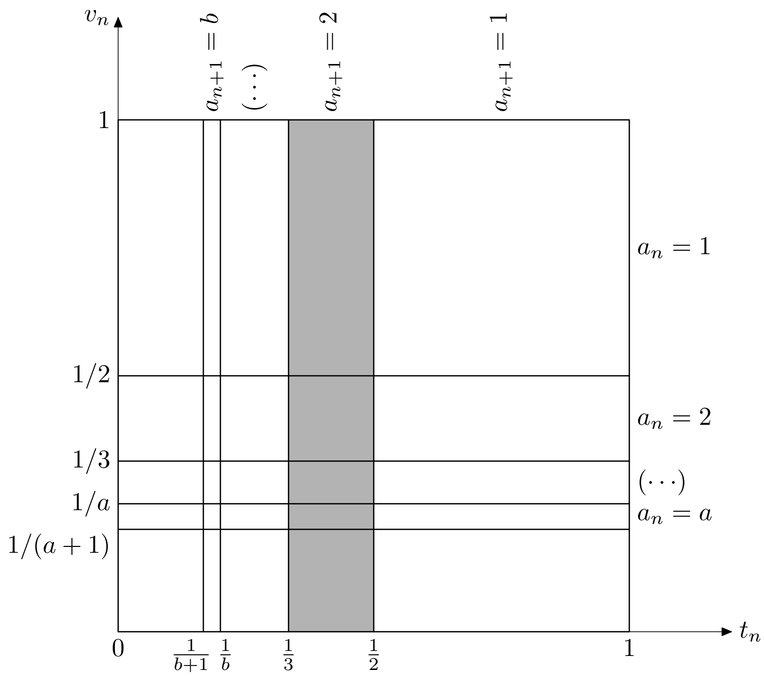


FIGURE 1. Strips in $\Omega = ([0,1)\setminus\mathbb{Q})\times[0,1]$. On a horizontal strip the digit a_n is constant and on each vertical strip a_{n+1} is constant. For instance, the gray strip contains all points (t_n, v_n) with $a_{n+1} = 2$.

3.1. Ergodicity. In an ergodic system the time average is related to the space average. Heuristically an ergodic dynamical system can not be seen as the union of two separate systems. Ergodicity is defined for a dynamical system $(X,\mathcal{F},\mu,T)$, where X is a non-empty set, $\mathcal{F}$ is an σ-algebra on X, μ is a probability measure on $(X,\mathcal{F})$ and $T: X\to X$ is a surjective μ-measure preserving transformation. If $\mathcal{F}$ is the Borel algebra of X we write (X,μ,T) instead of $(X,\mathcal{F},\mu,T)$.

Definition I.16. Let $(X,\mathcal{F},\mu,T)$ be a dynamical system. Then T is called *ergodic* if for every μ-measurable, T-invariant set A one has $\mu(A)=0$ or $\mu(A)=1$. ◇

The following theorem was obtained in 1977 by Nakada *et al.* [**43**]; see also [**41**] and [**20**].

Theorem I.17. *Let ν be the probability measure on Ω with density $d(x,y)$, given by*

$$d(x,y) = \frac{1}{\log 2}\,\frac{1}{(1+xy)^2}, \quad (x,y) \in \Omega. \tag{I.18}$$

Then, the dynamical system $(\Omega, \nu, \mathcal{T})$ is an ergodic system.

The system $(\Omega, \nu, \mathcal{T})$ is called the *natural extension* of the ergodic dynamical system $([0,1), \mu, T)$, where μ is the so-called Gauss-measure, the probability measure on $[0,1)$ with density

$$d(x) = \frac{1}{\log 2}\,\frac{1}{1+x}, \quad x \in [0,1).$$

Later we derive natural extensions for other continued fractions. In general, a natural extension is the smallest invertible dynamical system, that has the dynamics of the original transformation as a subsystem. Rohlin [**49**] introduced the concept of a natural extension of a non-invertible system in 1964. He showed that a natural extension is unique up to isomorphism, and proved that it possesses similar dynamical properties as the original system.

The ergodicity of $\mathcal{T}$ allows us to apply Birkhoff's ergodic theorem.

Theorem I.19. *Let $(X, \mathcal{F}, \nu, \mathcal{T})$ be a dynamical system with $\mathcal{T}$ an ergodic transformation. Then for any function f in $L^1(\mu)$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} f(\mathcal{T}^i(x)) = \int_X f \,\mathrm{d}\mu.$$

This is one of the main results in ergodic theory, see Chapter 3 of [**10**]. The following result on the distribution of the sequence $(t_n, v_n)_{n\geq 0}$ is a consequence of Birkhoff's ergodic theorem, and was obtained by Bosma *et al.* in [**4**]; see also Chapter 4 of [**10**].

Theorem I.20. *For almost all $x \in [0,1)$ the two-dimensional sequence*

$$(t_n, v_n) = \mathcal{T}^n(x, 0), \quad n \geq 0,$$

is distributed over Ω according to the density-function $d(t,v)$, as given in (I.18).

Corollary I.21. *Let $B \subset \Omega$ be a Borel measurable set with a boundary of Lebesgue measure zero. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I_B(t_n, v_n) = \nu(B), \tag{I.22}$$

where ν is as given in Theorem I.17 and I_B denotes the indicator function

$$I_B(t_n, v_n) = \begin{cases} 1 & \text{if } (t_n, v_n) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

We use this corollary in Chapter II to compute the probability that certain approximation coefficients are smaller than given values.

3.2. Entropy. Entropy is an important concept of information in physics, chemistry, and information theory. It can be seen as a measure for the amount of "disorder" of a system. Entropy also plays an important role in ergodic theory. Ornstein proved in 1968 that any two Bernoulli schemes (generalizations of the Bernoulli process to more than two possible outcomes) with the same entropy are isomorphic [**46**]; see also [**25**] or Chapter 1 of [**54**]. Like Birkhoff's ergodic theorem this is a fundamental result in ergodic theory.

For a measure preserving transformation the entropy is often defined by partitions, but in 1964 Rohlin [**49**] showed that the entropy of a μ-measure preserving operator $T : [a, b] \to [a, b]$ is given by the beautiful formula

$$h(T) = \int_a^b \log |T'(x)| \mathrm{d}\mu(x).$$

From Rohlin's formula it follows that the entropy of the RCF-operator is given by

$$h(T) = \int_0^1 \log |T'(x)| \mathrm{d}\mu(x) = -2 \int_0^1 \log(x) \mathrm{d}\mu(x) = \frac{-2}{\log 2} \int_0^1 \frac{\log(x)}{x+1} \mathrm{d}x = \frac{\pi^2}{6 \log 2}.$$

The following results are very useful to find the entropy of other operators; see Chapter 6 of [**10**].

Theorem I.23. *A measure preserving transformation has the same entropy as its natural extension.*

Definition I.24. Two dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(X', \mathcal{F}', \mu', T')$ are isomorphic if there exist measurable sets $N \subset X$ and $N' \subset X'$ with $\mu(X \setminus N) = \mu'(X' \setminus N') = 0, T(N) \subset N, T'(N') \subset N'$ and a measurable map $\psi : N \to N'$ such that

(1) ψ is one-to-one and onto almost everywhere,
(2) ψ is measure preserving,
(3) ψ preservers the dynamics of T and T', i.e. $\psi \circ T = S \circ \psi$.

Theorem I.25. *If two dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(X', \mathcal{F}', \mu', T')$ are isomorphic, then they have the same entropy.*

In Chapter IV we use these theorems to derive the entropy of a specific natural extension from an isomorphic dynamical system.

3.3. Asymmetric Diophantine approximation. In Chapter II we use the natural extension to study another quality measure for the approximations of RCF convergents. The inequality (I.7) can be strengthened to

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}, \qquad \text{for } n \geq 0.$$

For any irrational x we define the sequence C_n, $n \geq 0$ by

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{C_n q_n q_{n+1}}, \qquad \text{for } n \geq 0, \tag{I.26}$$

Tong derived in [**57**] and [**58**] various properties of the sequence $(C_n)_{n\geq 0}$, and of the related sequence $(D_n)_{n\geq 0}$, defined by

$$D_n = [a_{n+1}; a_n, \ldots, a_1] \cdot [a_{n+2}; a_{n+3}, \ldots] = \frac{1}{C_n - 1}, \qquad \text{for } n \geq 0. \tag{I.27}$$

For good approximations C_n is large and D_n small.

Remark I.28. Note that $D_n \in \mathbb{R} \setminus \mathbb{Q}$ and not just in $[0,1) \setminus \mathbb{Q}$. ◇

In Chapter II we focus on C_n and D_n as measures of approximation quality for regular continued fractions. Suppose the $n-1$-st approximation and the $n+1$-st approximation are both good, what can we say about the n-th approximation sandwiched between those two? Using the natural extension we prove the following theorem.

Theorem I.29. *Let $x = [\,a_0; a_1,\, a_2, \ldots,\, a_n, \ldots]$ be an irrational number, let $r, R > 1$ be reals and put*

$$\begin{aligned} F &= \frac{r(a_{n+1}+1)}{a_n(a_{n+1}+1)(r+1)+1}, \\ G &= \frac{R(a_n+1)}{(a_n+1)a_{n+1}(R+1)+1} \quad \textit{and} \\ M &= \frac{1}{2}\left(\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\left(1+\frac{1}{r}\right)\left(1+\frac{1}{R}\right)\right. \\ &\quad \left. + \sqrt{\left[\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\left(1+\frac{1}{r}\right)\left(1+\frac{1}{R}\right)\right]^2 - \frac{4}{rR}}\right). \end{aligned}$$

Assume $D_{n-2} < r$ and $D_n < R$.

(1) *If $r - a_n \geq G$ and $R - a_{n+1} < F$, then*
$$D_{n-1} > \frac{a_{n+1}+1}{R - a_{n+1}}.$$
(2) *If $r - a_n < G$ and $R - a_{n+1} \geq F$, then*
$$D_{n-1} > \frac{a_n+1}{r - a_n}.$$
(3) *In all other cases*
$$D_{n-1} > M.$$

These bounds are sharp. Furthermore, in case (1) $\frac{a_{n+1}+1}{R-a_{n+1}} > M$ and in case (2) $\frac{a_n+1}{r-a_n} > M$.

We prove a similar theorem for the case that $D_{n-2} > r$ and $D_n > R$ in Section II.3. In Section II.4 we calculate the asymptotic frequency that simultaneously $D_{n-2} > r$ and $D_n > R$. In Section II.5 we correct an incorrect result by Tong on C_n and give the sharp bounds for this case.

4. Other continued fractions

There are many different types of continued fractions. In this section we describe the nearest integer, α and (α)-Rosen fractions.

4.1. Nearest Integer Continued Fractions. The nearest integer continued fraction (NICF) operator rounds, as the name suggests, to the nearest integer.

Definition I.30. The NICF operator $f_{\frac{1}{2}} : [-\frac{1}{2}, \frac{1}{2}) \to [-\frac{1}{2}, \frac{1}{2})$ is defined by

$$f_{\frac{1}{2}}(x) = \frac{\varepsilon}{x} - \left\lfloor \frac{\varepsilon}{x} + \frac{1}{2} \right\rfloor \text{ if } x \neq 0 \text{ and } f_{\frac{1}{2}}(0) = 0,$$

◇

where ε denotes the sign of x. A NICF-expansion is denoted by

$$\frac{\varepsilon_1}{d_1 + \cfrac{\varepsilon_2}{d_2 + \cfrac{\varepsilon_3}{d_3 + \dots}}} = [\,\varepsilon_1 : d_1, \varepsilon_2 : d_2, \varepsilon_3 : d_3, \dots], \tag{I.31}$$

with $d_n \in \mathbb{N}, \varepsilon_n = \pm 1$ and $\varepsilon_{n+1} + d_n \geq 2$ for $n \geq 1$.

The ε_n and d_n are found by repeatedly applying $f_{\frac{1}{2}}$. Let $n \geq 1$ be such that $f_{\frac{1}{2}}^{n-1}(x) \neq 0$ (this is always true when x is irrational); then

$$\varepsilon_n = \operatorname{sgn}\left(f_{\frac{1}{2}}^{n-1}(x)\right) = \begin{cases} 1, & \text{if } f_{\frac{1}{2}}^{n-1}(x) > 0 \\ -1, & \text{if } f_{\frac{1}{2}}^{n-1}(x) < 0, \end{cases}$$

and

$$d_n = \left\lfloor \frac{\varepsilon_n}{f_{\frac{1}{2}}^{n-1}(x)} + \frac{1}{2} \right\rfloor.$$

We recycle notation and now write $p_n/q_n(x)$ for the nth NICF-convergent of x and the accompanying $\Theta_n(x)$ for the n-th approximation coefficient of x. Later we use this notation for other types of continued fractions as well.

In 1989, Jager and Kraaikamp [**23**] obtained a Borel result for the NICF.

Theorem I.32. *For every irrational x and all positive integers n one has*

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{5}{2}\left(5\sqrt{5} - 11\right).$$

The constant $\frac{5}{2}\left(5\sqrt{5} - 11\right)$ is best possible.

This result was extended by Tong in [**59**] and [**60**] as follows.

Theorem I.33. *For every irrational number x and all positive integers n and k one has*

$$\min\{\Theta_{n-1}, \Theta_n, \dots, \Theta_{n+k}\} < \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}.$$

The constant $\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}$ *is best possible.*

4.2. α**-expansions.** In 1907, McKinney [**38**] introduced α-expansions, a class of continued fractions generated by the operator f_α.

Definition I.34. Let $\frac{1}{2} \leq \alpha \leq 1$. The α-expansion operator $f_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined by

$$f_\alpha(x) = \frac{\varepsilon}{x} - \left\lfloor \frac{\varepsilon}{x} + 1 - \alpha \right\rfloor \text{ if } x \neq 0 \text{ and } f_\alpha(0) \;=\; 0,$$

where ε again denotes the sign of x. $\diamond$

For $\alpha = 1$ we find the RCF-expansion and for $\alpha = \frac{1}{2}$ the NICF-expansion. For any $\alpha \in \left[\frac{1}{2}, 1\right]$ and for every irrational x the α-continued fraction convergents form a subsequence of the RCF-convergents. In 1981, Nakada [**41**] determined the natural extension of the α-expansion operator and the entropy of T_α. In [**39**] Moussa *et al.* extended these results to the case $\sqrt{2} - 1 \leq \alpha < \frac{1}{2}$. More recently Luzzi and Marmi [**37**] and Nakada and Natsui [**44**] analysed the case $0 \leq \alpha < \sqrt{2} - 1$.

4.3. Rosen continued fractions. Rosen fractions were introduced in 1954 by David Rosen [**50**]. Let $q \in \mathbb{Z}, q \geq 3$ and $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$. For simplicity we usually write λ instead of λ_q. Notice that $\lambda \to 2$ if $q \to \infty$.

Definition I.35. For each q the Rosen-expansion operator $T_q : [-\frac{\lambda}{2}, \frac{\lambda}{2}) \to [-\frac{\lambda}{2}, \frac{\lambda}{2})$ is defined by

$$T_q(x) = \frac{\varepsilon}{x} - \lambda \left\lfloor \frac{\varepsilon}{\lambda x} + \frac{1}{2} \right\rfloor \text{ if } x \neq 0 \text{ and } T_q(0) \;=\; 0. \tag{I.36}$$

$\diamond$

Remark I.37. If $q = 3$, we have $\lambda = 1$ and we see that T_3 in (I.36) is the same as the NICF-operator $f_{\frac{1}{2}}$. $\diamond$

Signs and digits are found in a similar way as with the nearest integer continued fractions. A Rosen continued fraction has the form

$$\cfrac{\varepsilon_1}{d_1\lambda + \cfrac{\varepsilon_2}{d_2\lambda + \dots}} = [\; \varepsilon_1 : d_1, \varepsilon_2 : d_2, \dots, \;],$$

where $\varepsilon_i \in \{-1, +1\}$ and the d_i are positive integers.

Rosen defined his continued fractions in order to study aspects of the Hecke groups, $G_q \subset \mathrm{PSL}(2, \mathbb{R})$. We use the Möbius (or, fractional linear) action of 2×2 matrices on the reals (extended to include ∞, as necessary); see [**14**] for an excellent introduction to this subject.

Definition I.38. For a matrix A,

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a, b, c, d \in \mathbb{Z}$ and $\det A = ad - bc \in \{-1, +1\}$, we define (with slight abuse of notation) the Möbius transformation $A : \mathbb{C}^* \to \mathbb{C}^*$ by

$$A(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d} \, .$$

◇

Remark I.39. Note that A and $-A$ define the same Möbius transformation. We often use this. In particular we write

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

since

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}(x) = x.$$

◇

With fixed index q and $\lambda = \lambda_q$, set

$$S = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} \lambda & -1 \\ 1 & 0 \end{bmatrix} . \tag{I.40}$$

Then G_q is generated by any two of these, as $U = ST$. In fact, $U^q = \mathrm{Id}$, [**50**]. It follows that $U^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$ where the sequence B_n is given by

$$B_0 = 0, \quad B_1 = 1, \quad B_n = \lambda B_{n-1} - B_{n-2}, \quad \text{for } n = 2, 3, \dots . \tag{I.41}$$

We use the above extensively in Chapters III and IV.

4.4. α-Rosen continued fractions. Dajani *et al.* [**11**] introduced α-Rosen continued fractions, a generalization of both Nakada's α-fractions and Rosen continued fractions.

Definition I.42. Let λ be as before. For $\alpha \in \left[0, \frac{1}{\lambda}\right]$, put $\mathbb{I}_\alpha = [\,(\alpha-1)\lambda, \alpha\lambda\,)$. We define $T_\alpha : \mathbb{I}_\alpha \to \mathbb{I}_\alpha$ by

$$T_\alpha(x) = \frac{\varepsilon}{x} - \lambda \left\lfloor \frac{\varepsilon}{\lambda x} + 1 - \alpha \right\rfloor \text{ if } x \neq 0 \text{ and } T_\alpha(0) := 0. \tag{I.43}$$

◇

Remark I.44. Setting $q = 3$ gives Nakada's α-expansions from Definition I.34. Additionally setting $\alpha = 1$ gives the regular continued fractions and $\alpha = \frac{1}{2}$ the nearest integer continued fractions. On the other hand, fixing $\alpha = \frac{1}{2}$ gives the Rosen expansions. ◇

Remark I.45. We usually suppress the dependence on q in our notation when we are working with α-Rosen fractions. In the rest of this thesis if the subscript $*$ of T is an integer greater than 2, it denotes the Rosen map (I.36) with $q = *$, otherwise it denotes the α-Rosen map (I.43) with $\alpha = *$. ◇

4.4.1. *Borel and Hurwitz-results for α-Rosen fractions.* For simplicity, we say that a real number r/s is a G_q*-rational* if it has finite (α)-Rosen expansion, all other real numbers are called G_q*-irrationals*. In [**16**], Haas and Series derived a Hurwitz-type result using non-trivial hyperbolic geometric techniques. They showed that for every G_q-irrational x there exist infinitely many G_q-rationals r/s, such that $s^2 \left| x - \frac{r}{s} \right| \leq \mathcal{H}_q$, where $\mathcal{H}_q$ is given by

$$\mathcal{H}_q = \begin{cases} \dfrac{1}{2} & \text{if } q \text{ is even,} \\ \dfrac{1}{2\sqrt{(1-\frac{\lambda}{2})^2+1}} & \text{if } q \text{ is odd.} \end{cases}$$

In Chapter III we use a geometric method to generalize Borel's classical approximation results for the regular continued fraction expansion to the α-Rosen fraction expansion. This yields the α-Rosen counterpart of Theorem I.33. We use α-Rosen fractions to give a Haas-Series-type result about all possible good approximations for the α for which the Legendre constant is larger than the Hurwitz constant. Furthermore, we prove the following theorem.

Theorem I.46. *Let $\alpha \in [1/2, 1/\lambda]$ and denote the nth α-Rosen convergent by p_n/q_n. For every G_q-irrational x there are infinitely many $n \in \mathbb{N}$ for which*

$$q_n^2 \left| x - \frac{p_n}{q_n} \right| \leq \mathcal{H}_q.$$

The constant $\mathcal{H}_q$ is best possible.

In Section III.5 we determine the Legendre constant for odd α-Rosen fractions and extend the Borel-result to a Hurwitz-result for specific values of α.

4.4.2. *Natural extensions for α-Rosen fractions.* In [**11**] the domain Ω_α of the natural extension of T_α was derived for $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$. Recall that "the domain of the natural extension of T_α" refers to the largest region on which $\mathcal{T}_\alpha(x,y)$ is bijective almost surely.

The entropy of the α-Rosen map is given by

$$\int_{(\alpha-1)\lambda}^{\alpha\lambda} \log |T'_\alpha| \psi_\alpha(x) \, \mathrm{d}x,$$

where ψ_α is the probability density function for the measure with respect to which T_α is ergodic; see Chapter 10 of [**18**] and [**11**].

In Chapter IV we derive the domain for natural extensions for α-Rosen continued fractions with $\alpha < \frac{1}{2}$. We do this by appropriately adding and deleting rectangles from the region of the natural extension for the standard Rosen fractions. We prove the following result; also see Figure 2.

Theorem I.47. *Fix $q \in \mathbb{Z}, q \geq 4$ and $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$.*

(i.) *Let*

$$\alpha_0 = \begin{cases} \dfrac{\lambda^2 - 4 + \sqrt{\lambda^4 - 4\lambda^2 + 16}}{2\lambda^2} & \text{if } q \text{ is even,} \\[2ex] \dfrac{\lambda - 2 + \sqrt{2\lambda^2 - 4\lambda + 4}}{\lambda^2} & \text{otherwise.} \end{cases}$$

Then $(\,\alpha_0, 1/\lambda\,]$ *is the largest interval containing* $1/2$ *for which each domain of the natural extension of* T_α *is connected.*

(ii.) *Furthermore, let*

$$\omega_0 = \begin{cases} 1/\lambda & \text{if } q \text{ is even,} \\ \dfrac{\lambda - 2 + \sqrt{\lambda^2 - 4\lambda + 8}}{2\lambda} & \text{otherwise.} \end{cases}$$

Then the entropy of the α*-Rosen map for each* $\alpha \in [\,\alpha_0, \omega_0\,]$ *is equal to the entropy of the standard Rosen map.*

Remark I.48. The value of the entropy of the standard Rosen map was found by H. Nakada [**42**] to be $C \cdot \dfrac{(q-2)\pi^2}{2q}$, where C is the normalizing constant (which depends on the parity of the index q) given in [**8**]. ◇

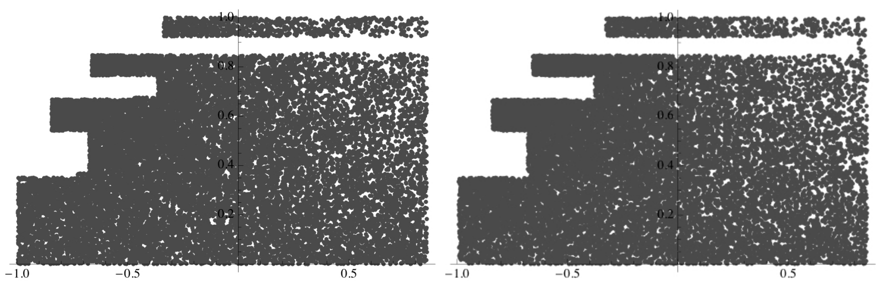


FIGURE 2. Simulations of the natural extension for $q = 8$ with on the x-axis $[(\alpha-1)\lambda, \alpha\lambda)$. On the left $\alpha = \alpha_0 - 0.001$ and on the right $\alpha = \alpha_0 + 0.001$. For $\alpha < \alpha_0$ the domain of the natural extension is disconnected.

In Section IV.2 we sketch the main argument of our approach — when the orbits of these basic regions agree after the same number of steps, entropy is preserved. We give an example of our techniques in Section IV.3 by re-establishing known results for certain classical Nakada α-fractions. In Sections IV.4 and IV.5 we give the proof of Theorem I.47, in the even and odd index case, respectively. Finally, in Section IV.6 we indicate how our results can be extended to show that in the odd index case, the entropy of T_α decreases when $\alpha > \omega_0$.

5. Multi-dimensional continued fractions

In 1842, Dirichlet [**35**] (Chapter XXXV of the first volume) proved that for every $a \in \mathbb{R} \setminus \mathbb{Q}$ there are infinitely many integers q such that

$$\|q\,a\| < q^{-1}, \tag{I.49}$$

where $\|x\|$ denotes the distance between x and the nearest integer. The exponent -1 of q cannot be replaced by a smaller number. It follows from (I.7) that the regular continued fraction algorithm generates an infinite sequence of fractions that satisfy this inequality.

As to the generalization of approximations in higher dimensions Dirichlet proved the following theorem; see Chapter II of [**51**].

Theorem I.50. *Let an $n \times m$ matrix A with entries $a_{ij} \in \mathbb{R} \backslash \mathbb{Q}$ be given and suppose that $1, a_{i1}, \ldots, a_{im}$ are linearly independent over $\mathbb{Q}$ for some i with $1 \leq i \leq n$. There exist infinitely many coprime m-tuples of integers $q_1, \ldots, q_m$ such that with $q = \max_j |q_j| \geq 1$, we have*

$$\max_i \|q_1\, a_{i1} + \cdots + q_m\, a_{im}\| < q^{\frac{-m}{n}}. \tag{I.51}$$

The exponent $\frac{-m}{n}$ of q is minimal.

If we take $m = n = 1$ this inequality is exactly (I.49). If we put $m = 1$, the result is about (simultaneous) Diophantine approximation: Given numbers $a_1, \ldots, a_n \in \mathbb{R}$, there is an integer q such that $\|q a_i\|$ is small compared to $q^{\frac{-1}{n}}$ for $i = 1, \ldots, n$. If we take $n = 1$, Theorem I.50 reduces to a statement about a linear combination with integer coefficients: given $a_1, \ldots, a_m \in \mathbb{R}$, there exist integers $q_1, \ldots, q_m$ such that $\|q_1 a_1 + \cdots + q_m a_m\| < q^{-m}$ where again $q = \max_j |q_j|$.

Definition I.52. Let an $n \times m$ matrix A with entries $a_{ij} \in \mathbb{R} \setminus \mathbb{Q}$ be given. The Dirichlet coefficient of an m-tuple $q_1, \ldots, q_m$ is defined as

$$q^{\frac{m}{n}} \max_i \|q_1\, a_{i1} + \cdots + q_m\, a_{im}\|.$$

◇

Remark I.53. The Dirichlet coefficient is a generalization of the approximation coefficient in the one-dimensional case. Notice that for $m = n = 1$ the Dirichlet quality equals Θ as defined in Definition I.9. ◇

For the case $m = 1$ the first multi-dimensional continued fraction algorithm was given by Jacobi [**22**]. Many more followed, see for instance Perron [**47**], Brun [**7**], Lagarias [**33**] and Just [**24**]. Brentjes [**6**] gives a detailed history and description of such algorithms. Schweiger's book [**53**] gives a broad overview. For $n = 1$ there is amongst others the algorithm by Ferguson and Forcade [**13**].

However, there is no efficient algorithm that is guaranteed to find a series of approximations with Dirichlet coefficient smaller than 1. In 1982, the LLL-algorithm for lattice basis reduction was published in [**36**]. Lenstra, Lenstra and Lovász noted that their algorithm could in polynomial time find Diophantine approximations of given rationals with Dirichlet coefficients only depending on the dimension. We first introduce lattices and then present the LLL-algorithm.

5.1. Lattices. Let r be a positive integer. A subset L of the r-dimensional real vector space $\mathbb{R}^r$ is called a *lattice* if there exists a basis $b_1, \ldots, b_r$ of $\mathbb{R}^r$ such that

$$L = \sum_{i=1}^{r} \mathbb{Z} b_i = \left\{ \sum_{i=1}^{r} z_i b_i ; z_i \in \mathbb{Z}\, (1 \leq i \leq r) \right\}.$$

We say that $b_1, \ldots, b_r$ is a *basis* for L. The *determinant* of the lattice L is defined by $\big| \det(b_1, \ldots, b_r) \big|$ and we denote it as $\det(L)$.

For any linearly independent $b_1, \ldots, b_r \in \mathbb{R}^r$ the Gram-Schmidt process yields an orthogonal basis $b_1^*, \ldots, b_r^*$ for $\mathbb{R}^r$. We define the orthogonal basis vectors inductively by

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{ij} b_j^* \quad \text{for } 1 \leq i \leq r \quad \text{and}$$

$$\mu_{ij} = \frac{(b_i, b_j^*)}{(b_j^*, b_j^*)},$$

where $(\,,)$ denotes the ordinary inner product on $\mathbb{R}^r$.

In most cases it is impossible to find a orthogonal basis for a lattice. A *reduced* basis for a lattice is a basis that consists of almost orthogonal vectors. In the original LLL-paper the following definition of a reduced lattice basis is used.

Definition I.54. A basis $b_1, \ldots, b_r$ for a lattice L is *reduced* if

$$|\mu_{ij}| \leq \frac{1}{2} \quad \text{for } 1 \leq j < i \leq r$$

and

$$|b_i^* + \mu_{ii-1} b_{i-1}^*|^2 \leq \frac{3}{4} |b_{i-1}^*|^2 \quad \text{for } 1 \leq i \leq r,$$

where $|x|$ denotes the Euclidean length of the vector x. ◇

The following properties of a reduced basis were shown in [**36**].

Proposition I.55. *Let $b_1, \ldots, b_r$ be a reduced basis for a lattice L in $\mathbb{R}^r$. Then we have*

$$\begin{aligned} &\text{(i)} && |b_1| \leq 2^{(r-1)/4} \big(\det(L)\big)^{1/r}, \\ &\text{(ii)} && |b_1|^2 \leq 2^{r-1}\, |x|^2, \quad \textit{for every } x \in L, x \neq 0, \\ &\text{(iii)} && \prod_{i=1}^{r} |b_i| \leq 2^{r(r-1)/4} \det(L). \end{aligned}$$

5.2. The LLL-algorithm. The LLL-algorithm finds a reduced basis for a given lattice in polynomial time. In each step the algorithm either swaps two successive basis vectors b_i and b_{i+1} or replaces b_i by $b_i - \lfloor \mu_{il} \rceil b_l$ for some index $l < i$. The main reasons the LLL-algorithm is fast are that only neighboring vectors are swapped and that vectors are only swapped if the swapping gives a progress bigger than a constant factor.

The original application of the LLL-algorithm was to give a polynomial time algorithm for factorizing polynomials with rational coefficients. The lattice basis reduction algorithm found many other applications in mathematics and computer science, in areas such as polynomial factorization, integer programming, and cryptology. For a description of the history of the LLL-algorithm and a survey of its applications, see [**45**].

The following proposition from [**36**] gives a bound for the number of arithmetic operations and for the integers on which they are performed.

Proposition I.56. *Let $L \subset \mathbb{Z}^r$ be a lattice with a basis $b_1, b_2, \ldots, b_r$, and let $F \in \mathbb{R}$, $F \geq 2$, be such that $|b_i|^2 \leq F$ for $1 \leq i \leq r$. Then the number of arithmetic operations used by the LLL-algorithm is $O(r^4 \log F)$ and the integers on which these operations are performed each have binary length $O(r \log F)$.*

In the following Lemma (which we prove in Chapter V) the approach suggested in the original LLL-paper for finding (simultaneous) Diophantine approximations is generalized to the case $m > 1$.

Lemma I.57. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$ and $\varepsilon \in (0,1)$ be given. Applying the LLL-algorithm to the basis formed by the columns of the $(m+n) \times (m+n)$-matrix*

$$B = \begin{bmatrix} 1 & 0 & \ldots & 0 & a_{11} & \ldots & a_{1m} \\ 0 & 1 & \ddots & 0 & a_{21} & \ldots & a_{2m} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 1 & a_{n1} & \ldots & a_{nm} \\ 0 & \ldots & 0 & 0 & c & & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \\ 0 & \ldots & 0 & 0 & 0 & & c \end{bmatrix},$$

with $c = \left(2^{-\frac{m+n-1}{4}} \varepsilon\right)^{\frac{m+n}{m}}$ yields an m-tuple $q_1, \ldots, q_m \in \mathbb{Q}$ with

$$\begin{aligned} \max_j |q_j| &\leq 2^{\frac{(m+n-1)(m+n)}{4m}} \varepsilon^{\frac{-n}{m}} \quad \textit{and} \\ \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| &\leq \varepsilon. \end{aligned}$$

It follows that the found m-tuple satisfies

$$\max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \leq 2^{\frac{(m+n-1)(m+n)}{4n}} q^{\frac{-m}{n}}, \tag{I.58}$$

where $q = \max_j |q_j|$, so the approximation has a Dirichlet coefficient of at most $2^{\frac{(m+n-1)(m+n)}{4n}}$.

5.3. The iterated LLL-algorithm. In Chapter V we present a multidimensional continued fraction algorithm that finds a sequence of approximations with Dirichlet coefficient only depending on the dimensions. This so-called Iterated LLL-algorithm (ILLL) repeatedly uses the LLL-algorithm for lattice basis reduction. We prove the following results.

Theorem I.59. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$, and $q_{\max} > 1$ be given. The ILLL algorithm finds a sequence of m-tuples $q_1, \ldots, q_m$ such that for every Q with $2^{\frac{(m+n+3)(m+n)}{4m}} \le Q \le q_{\max}$ one of these m-tuples satisfies*

$$\max_j |q_j| \le Q \text{ and}$$

$$\max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \le 2^{\frac{(m+n+3)(m+n)}{4n}} Q^{\frac{-m}{n}}.$$

Theorem I.60. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$ and $q_{\max} > 1$ be given. Assume that γ is such that for the Dirichlet coefficient of every m-tuple $q_1, \ldots, q_m$ returned by the ILLL algorithm one has*

$$q^{\frac{m}{n}} \max_i \|q_1 a_{i1} + \ldots q_m a_{im}\| \ge \gamma, \text{ where } q = \max_j |q_j|.$$

Put

$$\delta = 2^{\frac{-(m+n)^2(m+2n}{4n^2}} m^{\frac{-m}{2n}} n^{\frac{-1}{2}} \gamma^{\frac{m+n}{n}}. \tag{I.61}$$

Then every m-tuple $s_1, \ldots, s_m$ with

$$s = \max_j |s_j| < 2^{-\frac{(m+n+3)m+4n}{4m}} \left(\frac{n\delta^2}{m}\right)^{\frac{n}{2(m+n)}} q_{\max}$$

satisfies

$$s^{\frac{m}{n}} \max_i \|s_1 a_{i1} + \cdots + s_m a_{im}\| > \delta.$$

In Section V.4 we present a version of the algorithm that uses only rational numbers and prove that this modified algorithm runs in polynomial time of the input. In Section V.5 we present some experimental results obtained with the ILLL-algorithm.

II
Sharp bounds for symmetric and asymmetric Diophantine approximation

In the introduction we mentioned Borel's Theorem I.10 which states that, for every irrational number x and every $n \geq 1$,

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{5}}, \quad \text{where the constant } 1/\sqrt{5} \text{ is best possible.}$$

Over the last century this result has been refined in various ways. For example, in [**15**], [**40**], and [**2**], it was shown that

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{\sqrt{a_{n+1}^2 + 4}}, \qquad \text{for } n \geq 0,$$

while J.C. Tong showed in [**55**] that the "conjugate property" holds

$$\max\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} > \frac{1}{\sqrt{a_{n+1}^2 + 4}}, \qquad \text{for } n \geq 0.$$

Also various other results on Diophantine approximation have been obtained, starting with Dirichlet's observation from [**35**], that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}, \qquad \text{for } n \geq 0,$$

which lead to various results in symmetric and asymmetric Diophantine approximation; see e.g. [**56**], [**57**], [**27**], and [**28**].

Define for x irrational the number C_n by

$$x - \frac{p_n}{q_n} = \frac{(-1)^n}{C_n q_n q_{n+1}}, \qquad \text{for } n \geq 0. \tag{II.1}$$

Tong derived in [**57**] and [**58**] various properties of the sequence $(C_n)_{n\geq 0}$, and of the related sequence $(D_n)_{n\geq 0}$, where

$$D_n = [a_{n+1}; a_n, \ldots, a_1] \cdot [a_{n+2}; a_{n+3}, \ldots] = \frac{1}{C_n - 1}, \qquad \text{for } n \geq 0. \tag{II.2}$$

Remark II.3. Note that $D_n \in \mathbb{R} \setminus \mathbb{Q}$ and not just in $[0, 1) \setminus \mathbb{Q}$. In this chapter we assume $x \in \mathbb{R} \setminus \mathbb{Q}$ and we use the notation $x = [\, a_0; a_1,\, a_2, \ldots,\, a_n, \ldots]$. $\diamond$

Recently, Tong [**61**] obtained the following theorem, which covers many previous results.

Theorem II.4. (Tong) *Let* $x = [a_0; a_1, a_2, \dots, a_n, \dots]$ *be an irrational number. If* $r > 1$ *and* $R > 1$ *are two real numbers and*

$$\begin{aligned} M_{\text{Tong}} = \ & \frac{1}{2}\left(\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{R}\right)\right. \\ & \left. + \sqrt{\left[\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{R}\right)\right]^2 - \frac{4}{rR}}\right), \end{aligned}$$

then

(1) $D_{n-2} < r$ *and* $D_n < R$ *imply* $D_{n-1} > M_{\text{Tong}}$;
(2) $D_{n-2} > r$ *and* $D_n > R$ *imply* $D_{n-1} < M_{\text{Tong}}$.

Tong derived a similar result for the sequence C_n, but it is incorrect. We state this result, give a counterexample and present a correct version of it in Section 5.

The outline of this chapter is as follows. We derive elementary properties of the sequence D_n in Section 1. In Section 2 we prove Theorem I.29 that gives a sharp lower bound for the minimum of D_{n-1} in case $D_{n-2} < r$ and $D_n < R$ for real numbers $r, R > 1$. We prove a similar theorem for the case that $D_{n-2} > r$ and $D_n > R$ in Section 3. In Section 4 we calculate the asymptotic frequency with which simultaneously $D_{n-2} > r$ and $D_n > R$. Finally we correct Tong's result for C_n in Section 5 and give the sharp bound in this case.

1. The natural extension

The domain of the natural extension for regular continued fractions is given by $\Omega = ([0,1) \setminus \mathbb{Q}) \times [0,1]$. We denote points in Ω by (t, v) in general and use (t_n, v_n) when we are considering the point as the future and past of a number x at time n.

Lemma II.5. *Let* $x = [a_0; a_1, a_2, \dots]$ *be in* $\mathbb{R} \setminus \mathbb{Q}$ *and* $n \geq 2$ *be an integer. The variables* D_{n-2}, D_{n-1} *and* D_n *can be expressed in terms of future* t_n*, past* v_n *and digits* a_n *and* a_{n+1} *by*

$$D_{n-2} = D_{n-2}(t_n, v_n) = \frac{(a_n + t_n)v_n}{1 - a_n v_n}, \tag{II.6}$$

$$D_{n-1} = D_{n-1}(t_n, v_n) = \frac{1}{t_n v_n}, \quad \text{and} \tag{II.7}$$

$$D_n = D_n(t_n, v_n) = \frac{(a_{n+1} + v_n)t_n}{1 - a_{n+1} t_n}. \tag{II.8}$$

Proof. The expression for D_{n-1} follows from the definition in (II.2).

$$\begin{aligned} D_{n-1} &= [a_n; a_{n-1}, \dots, a_1][a_{n+1}; a_{n+2}, \dots] \\ &= \frac{1}{[0; a_n, a_{n-1}, \dots, a_1][0; a_{n+1}, a_{n+2}, \dots]} = \frac{1}{v_n t_n}. \end{aligned}$$

It follows in a similar way that $D_n = \frac{1}{t_{n+1}}\frac{1}{v_{n+1}}$ and using

$$t_{n+1} = \frac{1}{t_n} - a_{n+1}$$
$$v_{n+1} = \frac{q_n}{q_{n+1}} = \frac{q_n}{a_{n+1}q_n + q_{n-1}} = \frac{1}{a_{n+1} + v_n}$$

we find (II.8). The formula for D_{n-2} can be derived in a similar way. □

Remark II.9. Of course, D_{n-2}, D_{n-1} and D_n also depend on x, but we suppress this dependence in our notation. ◇

Using Theorem I.20 and its corollary we derive the following result.

Proposition II.10. *For almost all $x \in [0, 1)$, and for all $R \geq 1$, the limit*

$$\lim_{n\to\infty} \frac{1}{n}\#\{1 \leq j \leq n \,|\, D_j(x) \leq R\}$$

exists, and equals

$$H(R) = 1 - \frac{1}{\log 2}\left(\log\left(\frac{R+1}{R}\right) + \frac{\log R}{R+1}\right). \tag{II.11}$$

Consequently, for almost all $x \in [0, 1)$ one has that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} D_n(x) = \infty.$$

Proof. By (II.7) and Corollary I.21, for almost every x the asymptotic frequency with which $D_{n-1} \leq R$ is given by the measure of those points (t, v) in Ω with $\frac{1}{tv} \leq R$. This measure equals

$$\frac{1}{\log 2}\int_{t=\frac{1}{R}}^{1}\int_{v=\frac{1}{Rt}}^{1} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2};$$

also see Figure 1.

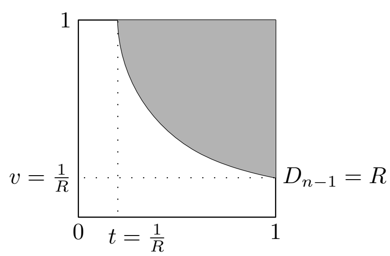


Figure 1. The curve $\frac{1}{tv} = R$ on Ω. For (t_n, v_n) in the gray part it holds that $D_{n-1} \leq R$.

It follows that

$$H(R) = \frac{1}{\log 2}\int_{\frac{1}{R}}^{1}\left[\frac{v}{1+tv}\right]_{\frac{1}{Rt}}^{1} \mathrm{d}t = \frac{1}{\log 2}\left[\log 2 - \log\frac{R+1}{R} - \frac{1}{R+1}\log R\right],$$

which may be rewritten as (II.11).

To calculate the expectation of D_n we use that the density function of D_n is given by $h(x) = H'(x)$, so

$$h(x) = \frac{1}{\log 2} \frac{\log x}{(x+1)^2}, \quad \text{for } x \geq 1.$$

We can now easily calculate the expected value of D_n

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} D_j(x) = \int_1^\infty x\, h(x)\, \mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{1}{\log 2} \frac{x \log x}{(x+1)^2}\, \mathrm{d}x = \infty.$$

$\square$

Besides for proving metric results on the D_n's, the natural extension $(\Omega, \nu, \mathcal{T})$ is also very handy to obtain various Borel-type results on the D_n's.

For $a, b \in \mathbb{N}$ consider the rectangle $\Delta_{a,b} = \left[\frac{1}{b+1}, \frac{1}{b}\right) \times \left[\frac{1}{a+1}, \frac{1}{a}\right) \subset \Omega$. On this rectangle we have $a_n = a$ and $a_{n+1} = b$. So $(t_n, v_n) \in \Delta_{a,b}$ if and only if $a_n = a$ and $a_{n+1} = b$. We use a and b as abbreviation for a_n and a_{n+1}, respectively, if we are working in such a rectangle.

We define two functions from $\left[\frac{1}{b+1}, \frac{1}{b}\right)$ to $\mathbb{R}$,

$$f_{a,r}(t) = \frac{r}{a(r+1)+t} \quad \text{and} \quad g_{b,R}(t) = \frac{R}{t} - b(R+1). \tag{II.12}$$

From (II.6) and (II.8) it follows for $(t_n, v_n) \in \Delta_{a,b}$ that

$$\begin{aligned} D_{n-2} < r \quad &\text{if and only if} \quad v_n < f_{a,r}(t_n), \\ D_n < R \quad &\text{if and only if} \quad v_n < g_{b,R}(t_n). \end{aligned}$$

We introduce the following notation

$$F = \frac{r(b+1)}{a(b+1)(r+1)+1} \quad \text{and} \quad G = \frac{R(a+1)}{(a+1)b(R+1)+1}. \tag{II.13}$$

We have that $F = f_{a,r}\left(\frac{1}{b+1}\right)$ and $g_{b,R}(G) = \frac{1}{a+1}$; also see Figure 2.

Remark II.14. The position of the graph of $f_{a,r}$ in $\Delta_{a,b}$ depends on a and r. Obviously we always have $f_{a,r}\left(\frac{1}{b}\right) < f_{a,r}\left(\frac{1}{b+1}\right) = F < \frac{1}{a}$. Furthermore

$$\begin{aligned} f_{a,r}\left(\frac{1}{b+1}\right) \geq \frac{1}{a+1} \quad &\text{if and only if} \quad r \geq a + \frac{1}{b+1}, \\ f_{a,r}\left(\frac{1}{b}\right) \geq \frac{1}{a+1} \quad &\text{if and only if} \quad r \geq a + \frac{1}{b}. \end{aligned}$$

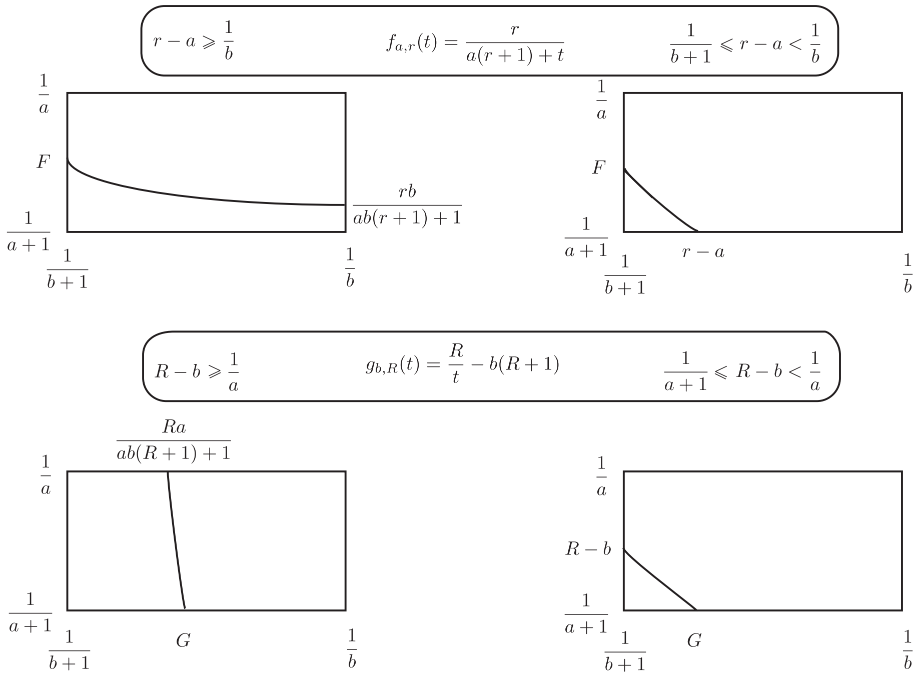


FIGURE 2. The possible intersection points of the graphs of $f_{a,r}$ and $g_{b,R}$ and the boundary of the rectangle $\Delta_{a,b}$, where $a_n = a$ and $a_{n+1} = b$.

Similarly, the position of the graph of $g_{b,R}$ in $\Delta_{a,b}$ depends on b and R. We always have $G < \frac{1}{b}$. Furthermore

$$\begin{aligned} G \geq \frac{1}{b+1} \quad &\text{if and only if} \quad R \geq b + \frac{1}{a+1}, \\ g_{b,R}\left(\frac{1}{b+1}\right) < \frac{1}{a} \quad &\text{if and only if} \quad R < b + \frac{1}{a}, \\ g_{b,R}\left(\frac{1}{b+1}\right) \geq \frac{1}{a+1} \quad &\text{if and only if} \quad R \geq b + \frac{1}{a+1}. \end{aligned}$$

Compare with Figure 2.

◇

We use the following lemma to determine where D_{n-1} attains it extreme values.

Lemma II.15. *Let $a, b \in \mathbb{N}$, and let $D_{n-1}(t,v) = \frac{1}{tv}$ for points $(t,v) \in (0,1] \times (0,1]$.*

(1) *When t is constant, D_{n-1} is monotonically decreasing as a function of v.*
(2) *When v is constant, D_{n-1} is monotonically decreasing as a function of t.*
(3) *$D_{n-1}(t,v)$ is monotonically decreasing as a function of t on the graph of $f_{a,r}$.*
(4) *$D_{n-1}(t,v)$ is monotonically increasing as a function of t on the graph of $g_{b,R}$.*

Proof. The first two statements follow from the trivial observation

$$\frac{\partial D_{n-1}}{\partial t} < 0 \quad \text{and} \quad \frac{\partial D_{n-1}}{\partial v} < 0. \tag{II.16}$$

For points (t,v) on the graph of $f_{a,r}$ we find $D_{n-1}(t,v) = \frac{a(r+1)+t}{rt}$ and

$$\frac{\partial D_{n-1}}{\partial t} = \frac{-a(r+1)}{rt^2} < 0,$$

which proves (3).

Finally, for points (t,v) on the graph of $g_{b,R}$ we find $D_{n-1}(t,v) = \frac{1}{R-b(R+1)t}$. So $\frac{\partial D_{n-1}}{\partial t} > 0$ and (4) is proven. □

Corollary II.17. *On $\Delta_{a,b}$ the infimum of D_{n-1} is attained in the upper right corner and its maximum in the lower left corner. To be more precise*

$$ab < D_{n-1} \leq (a+1)(b+1).$$

Lemma II.18. *Let $a,b \in \mathbb{N}$, $r,R > 1$, and set*

$$L = ab(r+1)(R+1),\ w = \sqrt{4LR + (r-R+L)^2)} \text{ and } S = \left(\frac{-L+R-r+w}{2b(R+1)}\right).$$

On $\mathbb{R}_+$ the graphs of $f_{a,r}$ and $g_{b,R}$ have one intersection point, which is given by

$$(S, f_{a,r}(S)) = \left(\frac{-L+R-r+w}{2b(R+1)}, \frac{2br(R+1)}{L+R-r+w}\right),$$

The corresponding value for D_{n-1} in this point is given by M_{Tong} as defined in Theorem II.4. For $x < S$ one has that $f_{a,r}(x) < g_{b,R}(x)$, while $f_{a,r}(x) > g_{b,R}(x)$ if $x > S$.

Proof. Solving

$$\frac{r}{a(r+1)+t} = \frac{R}{t} - b(R+1)$$

yields

$$S = \frac{-L+R-r+w}{2b(R+1)} \quad \text{or} \quad S = \frac{-L+R-r-w}{2b(R+1)}.$$

Since $L > R$ the second solution is always negative, so this solution cannot be in $\Delta_{a,b}$. The second coordinate follows from substituting $S = \frac{-L+R-r+w}{2b(R+1)}$ in $f_{a,r}(t)$ or $g_{b,R}(t)$.

The corresponding value for D_{n-1} in this point is given by

$$\begin{aligned} &D_{n-1}\left(\frac{-L+R-r+w}{2b(R+1)}, \frac{2br(R+1)}{L+R-r+w}\right) = \frac{-L-R+r-w}{r(L-R+r-w)} \\ &= \frac{-L^2+r^2-2Rr+R^2-2Lw-w^2}{r((L-R+r)^2-w^2)} = \frac{-2L^2-2Lw-2Lr-2LR}{-4RrL} \\ &= \frac{1}{2}\left(\frac{1}{r}+\frac{1}{R}+\frac{L}{Rr}+\frac{w}{Rr}\right) = M_{\text{Tong}}. \end{aligned}$$

Since $\lim_{x\downarrow 0} f_{a,r}(x) = \dfrac{r}{a(r+1)}$ and $\lim_{x\downarrow 0} g_{b,R}(x) = \infty$, we immediately have that $f_{a,r}(x) < g_{b,R}(x)$ if $x < S$. And because there is only one intersection point on $\mathbb{R}_+$, it follows that $f_{a,r}(x) > g_{b,R}(x)$ if $x > S$. $\square$

Remark II.19. In view of Remark II.14 and the last statement of Lemma II.18 the only possible configurations for $f_{a,r}$ and $g_{b,R}$ in $\Delta_{a,b}$ are given in Figure 3. $\diamond$

2. The case $D_{n-2} < r$ and $D_n < R$

We assume that both D_{n-2} and D_n are smaller than some given reals r and R. We recall Theorem I.29 from the Introduction.

Theorem II.20. *Let $r, R > 1$ be reals, let $n \geq 1$ be an integer and let F and G be as given in* (II.13)*. Assume $D_{n-2} < r$ and $D_n < R$.*

(1) *If $r - a_n \geq G$ and $R - a_{n+1} < F$, then*
$$D_{n-1} > \frac{a_{n+1}+1}{R - a_{n+1}}.$$
(2) *If $r - a_n < G$ and $R - a_{n+1} \geq F$, then*
$$D_{n-1} > \frac{a_n + 1}{r - a_n}.$$
(3) *In all other cases*
$$D_{n-1} > M_{\mathrm{Tong}}.$$

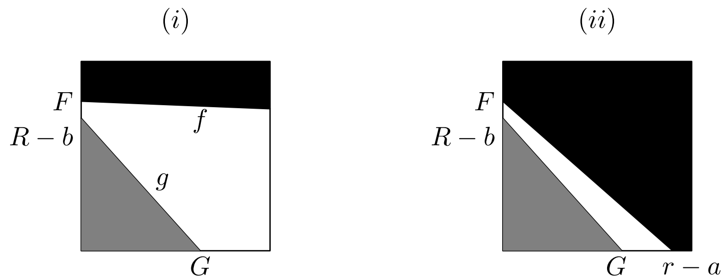
These bounds are sharp. Furthermore, in case (1) $\frac{a_{n+1}+1}{R-a_{n+1}} > M_{\mathrm{Tong}}$ *and in case* (2) $\frac{a_n+1}{r-a_n} > M_{\mathrm{Tong}}$.

Proof. We consider the closure of the region containing all points (t, v) in $\Delta_{a,b}$ with $D_{n-2}(t, v) < r$ and $D_n(t, v) < R$. In Figure 3 we show all possible configurations of this region.

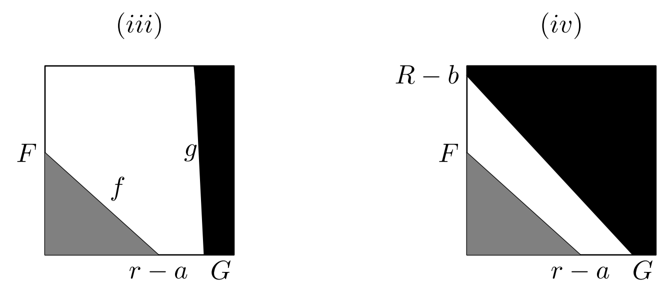
From (II.16) it follows that the extremum of D_{n+1} is attained in a boundary point. Lemma II.15 implies that we only need to consider the following three points

(1) The intersection point of the graph of $g_{b,R}$ and the line $t = \frac{1}{b+1}$, given by $\left(\frac{1}{b+1}, R - b\right)$.
(2) The intersection point of the graph of $f_{a,r}$ and the line $v = \frac{1}{a+1}$, given by $\left(r - a, \frac{1}{a+1}\right)$.
(3) The intersection point of the graphs of $f_{a,r}$ and $g_{b,R}$, given by M_{Tong}.

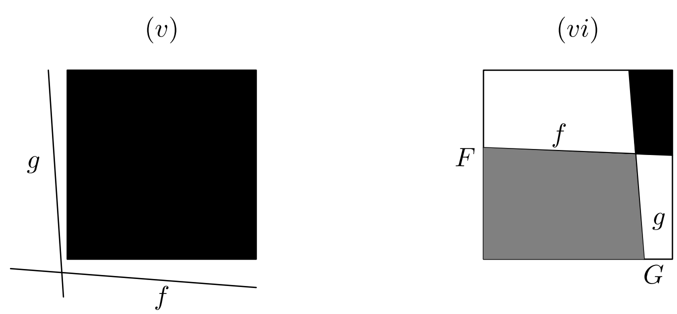
Assume $r - a \geq G$ and $R - b < F$. We know from Lemma II.18 that the graphs of $f_{a,r}$ and $g_{b,R}$ cannot intersect more than once in $\Delta_{a,b}$, thus we are in case (1); see Figure 3 (*i*) and (*ii*). In this case the minimum of D_{n-1} is given by $D_{n-1}\left(\frac{1}{b+1}, R - b\right) = \frac{b+1}{R-b}$.



(a) In case (i) and (ii) we have $r - a \geq G$ and $R - b < F$. It is allowed that $R - b < \frac{1}{a+1}$. In case (i) we have $r - a > \frac{1}{b}$ and in case (ii) $r - a \leq \frac{1}{b}$.



(b) In cases (iii) and (iv) we have $r - a < G$ and $R - b \geq F$. It is allowed that $r - a < \frac{1}{b+1}$. In case (iii) we have $R - b > \frac{1}{a}$ and in case (iv) $R - b \leq \frac{1}{a}$.



(c) In case (v) we have $F < \frac{1}{a+1}$ and $G < \frac{1}{b+1}$. Case (vi) contains all other cases, it can be separated in four subcases, see Figure 6.

FIGURE 3. The possible configurations of the graphs of $f_{a,r}$ and $g_{b,R}$ on $\Delta_{a,b}$, indicated by f and g, respectively. On the gray parts $D_{n-2} < r$ and $D_n < R$, on the black parts $D_{n-2} > r$ and $D_n > R$

Assume $r - a < G$ and $R - b \geq F$, then we are in case (2); see Figure 3 (*iii*) and (*iv*) and the minimum is given by $D_{n-1} = \left(r - a, \frac{1}{a+1}\right) = \frac{a+1}{r-a}$. A similar argument as before shows $M_{\mathrm{Tong}} < \frac{a+1}{r-a}$.

Otherwise, still assuming there are points $(t, v) \in \Delta_{a,b}$ with $D_{n-2}(t, v) < r$ and $D_n(t, v) < R$, we must be in case (3); see Figure 3 (vi). The minimum follows from Lemma II.18.

These bounds are sharp since the minimum is attained in the extreme point. $\square$

Example II.21. Take $r = 2.9$ and $R = 3.6$; see Figure 4.

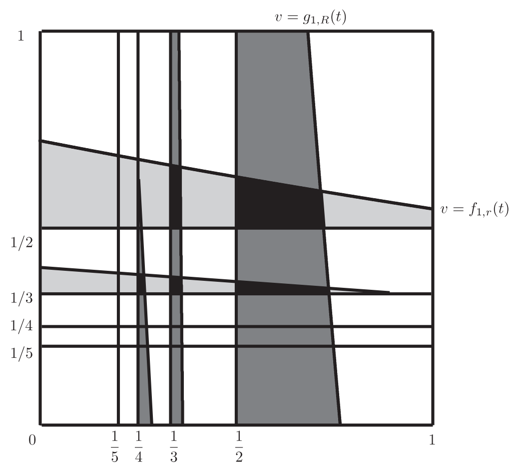


FIGURE 4. Example with $r = 2.9$ and $R = 3.6$. The regions where $D_{n-2} < 2.9$ are light gray, the regions where $D_n < 3.6$ are dark gray. The intersection where both $D_{n-2} < 2.9$ and $D_n < 3.6$ is black. The horizontal and vertical black lines are drawn to identify the strips and have no meaning for the value of D_{n-2} and D_n.

If $a_n = a_{n+1} = 1$, then $r - a_n = 1.9, R - a_{n+1} = 2.6, F \approx 0.66$ and $G \approx 0.71$. Since $R - a_{n+1} > F$ we do not have case (i) of Theorem II.20. Since $r - a_n > G$ we are not in case (ii) either. So in this case $D_{n-1} > M_{\text{Tong}} \approx 2.30$. For the following combinations the minimum is also given by M_{Tong}

$$\begin{aligned}
a_n = 1 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\text{Tong}} &\approx 4.04.\\
a_n = 2 \text{ and } a_{n+1} = 1 : D_{n-1} > M_{\text{Tong}} &\approx 4.04.\\
a_n = 2 \text{ and } a_{n+1} = 2 : D_{n-1} > M_{\text{Tong}} &\approx 7.48.\\
a_n = 2 \text{ and } a_{n+1} = 3 : D_{n-1} > M_{\text{Tong}} &\approx 10.92.
\end{aligned}$$

If $a_n = 1$ and $a_{n+1} = 3$, then $F \approx 0.70$ and $G \approx 0.25$. So $r - a_n > G$ and $\dfrac{1}{a_{n+1}} < R - a_{n+1} < F$. Thus

$$D_{n-1} > \frac{a_{n+1} + 1}{R - a_{n+1}} \approx 6.67 > M_{\text{Tong}} \approx 5.76.$$

For all other values of a_n and a_{n+1} either $D_{n-2} > r$ or $D_n > R$, or both.

$\diamond$

3. The case $D_{n-2} > r$ and $D_n > R$

In this section we study the case that D_{n-2} and D_n are larger than given reals r and R, respectively.

Theorem II.22. *Let $r, R > 1$ be reals, let $n \geq 1$ be an integer and let F and G be as given in* (II.13). *Assume $D_{n-2} > r$ and $D_n > R$.*

(1) *If $r - a_n \geq G$ and $R - a_{n+1} < F$, then*
$$D_{n-1} < \frac{a_{n+1}+1}{F}.$$
(2) *If $r - a_n < G$ and $R - a_{n+1} \geq F$, then*
$$D_{n-1} < \frac{a_n+1}{G}.$$
(3) *If $r - a_n < \frac{1}{a_{n+1}+1}$ and $R - a_{n+1} < \frac{1}{a_n+1}$, then*
$$D_{n-1} < (a_n+1)(a_{n+1}+1).$$
(4) *In all other cases*
$$D_{n-1} < M_{\text{Tong}}.$$

The bounds are sharp. Furthermore, in case (1) $\frac{a_{n+1}+1}{F} < M_{\text{Tong}}$, *in case* (2) $\frac{a_n+1}{G} < M_{\text{Tong}}$ *and in case* (3) $(a_n+1)(a_{n+1}+1) < M_{\text{Tong}}$.

Proof. The proof is very similar to that of Theorem II.20. The only 'new' case is the one where $r - a < \frac{1}{b+1}$ and $R - b < \frac{1}{a+1}$; see Figure 3 (v). If $r - a < \frac{1}{b+1}$, then the graph of $f_{a,r}$ lies below $\Delta_{a,b} \subset \Omega$. Similarly, if $R - b < \frac{1}{a+1}$ the graph $g_{b,R}$ lies left left of $\Delta_{a,b} \subset \Omega$. In this case we have that $D_{n-2} > r$ and $D_n > R$ for all $(t_n, v_n) \in \Delta_{a,b}$. In this case D_{n-1} attains its maximum in the lower left corner $\left(\frac{1}{b+1}, \frac{1}{a+1}\right)$. For the intersection point $(S, f_{a,r}(S))$ either $S < \frac{1}{b+1}$ or $f_{a,r}(S) < \frac{1}{a+1}$ and from Lemma II.15 we conclude $(a+1)(b+1) < M_{\text{Tong}}$. □

Example II.23. We again use $r = 2.9$ and $R = 3.6$; see Figure 5 and Table 1. ⋄

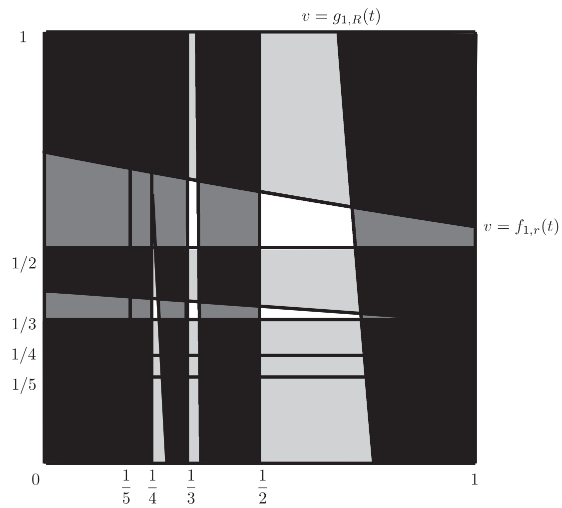


FIGURE 5. Example with $r = 2.9$ and $R = 3.6$. The regions where $D_{n-2} > 2.9$ are light gray, the regions where $D_n > 3.6$ are dark gray. The intersection where both $D_{n-2} > 2.9$ and $D_n > 3.6$ is black.

a_n	a_{n+1}	Case	Upper bound for D_{n-1}	Tong's upper bound
1	1	(vi_a)	2.30	2.30
1	2	(vi_a)	4.04	4.04
1	3	(i)	5.72	5.76
1	4	(i)	7.07	7.48
1	$5, 6, \ldots$	(i)	$\ldots$	$\ldots$
1	37	(i)	51.44	64.20
2	1	(vi_c)	4.04	4.04
2	2	(vi_a)	7.48	7.48
2	3	(vi_a)	10.92	10.92
2	4	(i)	13.79	14.36
2	$5, 6, \ldots$	(i)	$\ldots$	$\ldots$
2	42	(i)	116.00	144.97
3	1	(iii)	4.04	5.76
3	2	(iii)	7.48	10.92
3	3	(iii)	10.92	16.08
3	4	(v)	13.79	21.23
$4, 5, 6 \ldots$	$1, 2, 3$	(iii)	$\ldots$	$\ldots$
$3, 4, 5 \ldots$	$4, 5, 6, \ldots$	(v)	$\ldots$	$\ldots$
17	29	(v)	540.00	847.79

TABLE 1. The sharp upper bounds and the Tong bounds for D_{n-1} for $r = 2.9$ and $R = 3.6$. See Figure 3 for cases (i)-(v) and Figure 6 for (vi_a) and (vi_c).

4. Asymptotic frequencies

Due to Theorem I.17 and the ergodic theorem, the asymptotic frequency of an event is equal to the measure of the area of this event in the natural extension. We calculate the measure of the region where $D_{n-2} > r$ and $D_n > R$. The same calculations can be done in the easier case where $D_{n-2} < r$ and $D_n < R$.

4.1. The measure of the region where $D_{n-2} > r$ and $D_n > R$ in a rectangle $\Delta_{a,b}$. We calculate the measure in $\Delta_{a,b}$ above the graphs of $f_{a,r}$ and $g_{b,R}$ in the six cases from Figure 3. We denote $\log 2$ times the measure for case $(*)$ in $\Delta_{a,b}$ by $m_{a,b}^{(*)}$.

$$
\begin{aligned}
m_{a,b}^{(i)} &= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} = \int_{\frac{1}{b+1}}^{\frac{1}{b}} \left[\frac{-1}{t}\frac{1}{1+tv}\right]_{\frac{r}{a(r+1)+t}}^{\frac{1}{a}} \mathrm{d}t \\
&= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{-1}{t}\frac{a}{a+t} + \frac{1}{t}\frac{a(r+1)+t}{(a+t)(r+1)}\,\mathrm{d}t \\
&= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{-1}{t} + \frac{1}{a+t} + \frac{1}{t} - \frac{r}{(a+t)(r+1)}\,\mathrm{d}t \\
&= \int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{1}{(a+t)(r+1)}\,dt = \frac{1}{(r+1)}\big[\log(a+t)\big]_{\frac{1}{b+1}}^{\frac{1}{b}} \\
&= \frac{1}{(r+1)} \log\frac{(ab+1)(b+1)}{(ab+a+1)b}.
\end{aligned}
$$

Next we compute $m_{a,b}^{(v)}$, because it is handy for finding $m_{a,b}^{(ii)}$.

$$
m_{a,b}^{(v)} = \int_{\frac{1}{b+1}}^{\frac{1}{b}} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} = \log\frac{(ab+1)(ab+a+b+2)}{(ab+a+1)(ab+b+1)}.
$$

For $m_{a,b}^{(ii)}$ we subtract the measure of the region in $\Delta_{a,b}$ below the graph of $f_{a,r}$ from $m_{a,b}^{(v)}$.

$$
\begin{aligned}
m_{a,b}^{(ii)} &= m_{a,b}^{(v)} - \int_{\frac{1}{b+1}}^{r-a} \int_{\frac{1}{a+1}}^{f_{a,r}(t)} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} \\
&= \log\frac{(ab+1)(ab+a+b+2)}{(ab+b+1)(ab+a+1)} - \frac{r}{r+1}\log\frac{r(b+1)}{ab+a+1} - \log\frac{ab+a+b+2}{(b+1)(r+1)} \\
&= \log\frac{(ab+1)(b+1)(r+1)}{(ab+b+1)(ab+a+1)} - \frac{r}{r+1}\log\frac{r(b+1)}{ab+a+1}.
\end{aligned}
$$

In the computation of $m_{a,b}^{(iii)}$ we use that $v = g_{b,R}(t)$ if and only if $t = \frac{R}{v+b(R+1)}$, so

$$
m_{a,b}^{(iii)} = \int_{\frac{1}{a+1}}^{\frac{1}{a}} \int_{\frac{R}{b(R+1)+v}}^{\frac{1}{b}} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} = \frac{1}{(R+1)}\log\frac{(ab+1)(a+1)}{(ab+b+1)a}.
$$

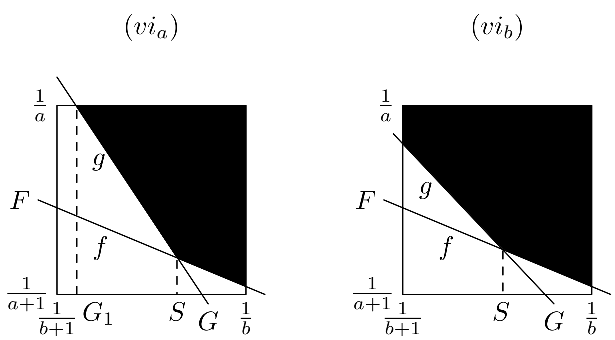
Note that $m_{a,b}^{(iii)}$ is $m_{a,b}^{(i)}$ with a interchanged with b and r replaced by R.

For $m_{a,b}^{(iv)}$ we find using the same techniques as before

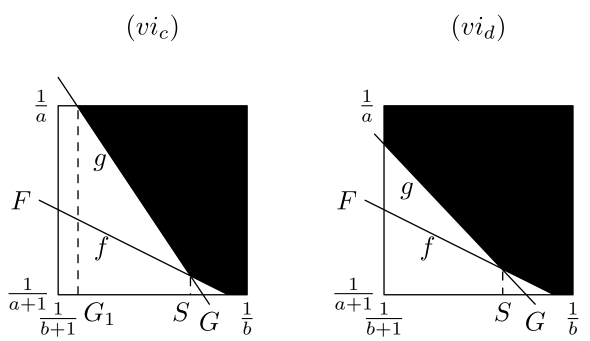
$$\begin{aligned} m_{a,b}^{(iv)} &= m_{a,b}^{(v)} - \int_{\frac{1}{a+1}}^{R-b} \int_{\frac{1}{b+1}}^{\frac{R}{b(R+1)+v}} \frac{\mathrm{d}t\,\mathrm{d}v}{(1+tv)^2} \\ &= \log \frac{(ab+1)(a+1)(R+1)}{(ab+a+1)(ab+b+1)} - \frac{R}{R+1} \log \frac{R(a+1)}{ab+b+1}, \end{aligned}$$

which is $m_{a,b}^{(ii)}$ where a is interchanged with b and r replaced by R.

In case (vi) there are four possibilities for the measure of the part above the graphs of $f_{a,r}$ and $g_{b,R}$, depending on where the graphs intersect with $\Delta_{a,b}$; see Figure 6. Denote $G_1 = \frac{Ra}{ab(R+1)+1}$ (found from solving $g_{b,R}(G_1) = \frac{1}{a}$) and recall from Lemma II.18 that S is the first coordinate of the intersection point of the graphs of $f_{a,r}$ and $g_{b,R}$. In this case we have that $(S, f_{a,r}(S)) \in \Delta_{a,b}$.



(a) In case (vi_a) we have $r - a \geq \frac{1}{b}$ and $R - b \geq \frac{1}{a}$ and in case (vi_b) we have $r - a \geq \frac{1}{b}$ and $R - b < \frac{1}{a}$.



(b) In case (vi_c) we have $r - a < \frac{1}{b}$ and $R - b \geq \frac{1}{a}$. In case (vi_d) we have $r - a < \frac{1}{b}$ and $R - b < \frac{1}{a}$.

FIGURE 6. The four possible configurations for case (vi).

(vi_a) If $r - a \geq \frac{1}{b}$ and $R - b \geq \frac{1}{a}$, then

$$m_{a,b}^{(vi_a)} = \int_{G_1}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{\frac{1}{b}} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

(vi_b) If $r-a \geq \frac{1}{b}$ and $R-b < \frac{1}{a}$, then

$$m_{a,b}^{(vi_b)} = \int_{\frac{1}{b+1}}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{\frac{1}{b}} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

(vi_c) If $r-a < \frac{1}{b}$ and $R-b \geq \frac{1}{a}$, then

$$m_{a,b}^{(vi_c)} = \int_{G_1}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{r-a} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{r-a}^{\frac{1}{b}} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

(vi_d) If $r-a < \frac{1}{b}$ and $R-b < \frac{1}{a}$, then

$$m_{a,b}^{(vi_d)} = \int_{\frac{1}{b+1}}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{S}^{r-a} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} + \int_{r-a}^{\frac{1}{b}} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}.$$

Using the following integrals

$$\begin{aligned}
\int_{x}^{S} \int_{g_{b,R}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} &= \frac{1}{R+1}\log\frac{S(1-bx)}{x(1-bS)} + \log\frac{x(S+a)}{S(x+a)},\\
\int_{S}^{y} \int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} &= \frac{1}{r+1}\log\frac{a+y}{a+S},\\
\int_{r-a}^{\frac{1}{b}} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2} &= \log\frac{(ab+1)(r+1)}{(ab+b+1)r},
\end{aligned}$$

we find that

$$\begin{aligned}
m_{a,b}^{(vi_a)} &= \frac{1}{R+1}\log\frac{S(1-bG_1)}{G_1(1-bS)} + \frac{1}{r+1}\log\frac{ab+1}{(a+S)b} + \log\frac{G_1(S+a)}{S(G_1+a)},\\
m_{a,b}^{(vi_b)} &= \frac{1}{R+1}\log\frac{S}{(1-bS)} + \frac{1}{r+1}\log\frac{ab+1}{(a+S)b} + \log\frac{S+a}{S(ab+a+1)},\\
m_{a,b}^{(vi_c)} &= \frac{1}{R+1}\log\frac{S(1-bG_1)}{G_1(1-bS)} + \frac{1}{r+1}\log\frac{r}{a+S} + \log\frac{G_1(S+a)(ab+1)(r+1)}{S(G_1+a)(ab+b+1)r},\\
m_{a,b}^{(vi_d)} &= \frac{1}{R+1}\log\frac{S}{(1-bS)} + \frac{1}{r+1}\log\frac{r}{a+S} + \log\frac{(S+a)(ab+1)(r+1)}{S(ab+a+1)(ab+b+1)r}.
\end{aligned}$$

4.2. The total measure of the region where $D_{n-2} > r$ and $D_n > R$ in Ω. For every $r > 1$ and $R > 1$ the asymptotic frequency with which $D_{n-2} > r$ and $D_n > R$ can be found by adding a finite number of integrals. Let $\{x\} = x - \lfloor x \rfloor$ and 1_A be the indicator function of A, i.e.

$$1_A = \begin{cases} 1 & \text{if condition } A \text{ is satisfied,} \\ 0 & \text{else.} \end{cases}$$

Proposition II.24. *For almost all $x \in [0,1)$, and for all $r, R \geq 1$, we have that*

$$\log 2 \lim_{n\to\infty} \frac{1}{n} \#\{2 \leq j \leq n+1; D_{j-2} > r \text{ and } D_j > R\}$$

exists and equals

$$
\begin{aligned}
& \sum_{a=1}^{\lfloor r \rfloor -1} \sum_{b=\lfloor R \rfloor +1}^{\infty} m^{(i)}_{a,b} \\
+ & \sum_{a=1}^{\lfloor r \rfloor -1} \left(1_{(\{R\} \le F)} m^{(i)}_{a,\lfloor R \rfloor} + 1_{(\{R\} \ge \frac{1}{a})} m^{(vi_a)}_{a,\lfloor R \rfloor} + 1_{(F < \{R\} < \frac{1}{a})} m^{(vi_b)}_{a,\lfloor R \rfloor} \right) \\
+ & \sum_{a=1}^{\lfloor r \rfloor -1} \sum_{b=1}^{\lfloor R \rfloor -1} m^{(vi_a)}_{a,b} + \sum_{b=\lfloor R \rfloor +1}^{\infty} \left(1_{(\{r\} \ge \frac{1}{b})} m^{(i)}_{\lfloor r \rfloor,b} + 1_{(\frac{1}{b+1} < \{r\} < \frac{1}{b})} m^{(ii)}_{\lfloor r \rfloor,b} \right) \\
+ & M_{r,R} + \sum_{b=1}^{\lfloor R \rfloor -1} \left(1_{(\{r\} \le G)} m^{(iii)}_{\lfloor r \rfloor,b} + 1_{(\{r\} \ge \frac{1}{b})} m^{(vi_a)}_{\lfloor r \rfloor,b} + 1_{(G < \{r\} < \frac{1}{b})} m^{(vi_c)}_{\lfloor r \rfloor,b} \right) \\
+ & \sum_{a=\lfloor r \rfloor +1}^{\infty} \sum_{b=\lfloor R \rfloor +1}^{\infty} m^{(v)}_{a,b} \\
+ & \sum_{a=\lfloor r \rfloor +1}^{\infty} \left(1_{(\{R\} \ge \frac{1}{a})} m^{(iii)}_{a,\lfloor R \rfloor} + 1_{(\{R\} \ge \frac{1}{a})} m^{(vi_a)}_{a,\lfloor R \rfloor} + 1_{(F < \{R\} < \frac{1}{a})} m^{(vi_b)}_{a,\lfloor R \rfloor} \right) \\
+ & \sum_{a=\lfloor r \rfloor +1}^{\infty} \sum_{b=1}^{\lfloor R \rfloor -1} m^{(iii)}_{a,b},
\end{aligned}
$$

where $M_{r,R}$ *is the measure of the regions where* $D_{n-2} > r$ *and* $D_n > R$ *in* $\Delta_{\lfloor r \rfloor, \lfloor R \rfloor}$.

PROOF. Let $a, b \ge 1$ be integers. We denote strips with constant a_n or a_{n+1} by

$$H_a = [0,1] \times \left[\frac{1}{a+1}, \frac{1}{a} \right] \quad \text{and} \quad V_b = \left[\frac{1}{b+1}, \frac{1}{b} \right] \times [0,1].$$

For $a < \lfloor r \rfloor$ the curve $v = f_{a,r}(t)$ is entirely inside the rectangle H_a and (depending on the position of the curve $v = g_{b,R}(t)$) we are either in case (i) or (vi); see Figure 3 and Remark II.14. If $a > \lfloor r \rfloor$ the curve $v = f_{a,r}(t)$ is entirely underneath H_a and we are in case (iii), (iv) or (v). For $a = \lfloor r \rfloor$ the curve $v = f_{a,r}(t)$ is partially inside and partially underneath $H_{\lfloor r \rfloor}$. In this strip we can have each of the six cases.

Similarly, for $b < \lfloor R \rfloor$, the curve of $v = g_{b,R}(t)$ is entirely inside the rectangle V_b and (depending on the position of the curve $v = g_{b,R}(t)$) we are in case (iii) or (vi). For $b > \lfloor R \rfloor$ the curve $v = g_{b,R}(t)$ is left of V_b and we are in case (i), (ii) or (v) . For $b = \lfloor R \rfloor$ the curve $v = g_{b,R}(t)$ is partially inside and partially left of $V_{\lfloor R \rfloor}$ and we can have each of the six cases.

We use the strips $H_{\lfloor r \rfloor}$ and $V_{\lfloor R \rfloor}$ to divide Ω in nine rectangles. Each of the nine terms in the sum in the proposition gives the measure of the region where $D_{n-2} > r$ and $D_n > R$ on one of those rectangles, we work from left to right and from top to bottom. The results follow from Corollary I.21, Remark II.14, Theorem II.22 and the above. For instance, the first rectangle is given by $\left[0, \frac{1}{\lfloor R+1 \rfloor}\right) \times \left[\frac{1}{\lfloor r \rfloor}, 1\right)$ and we see that for every $\Delta_{a,b}$ in this rectangle we are in case (i). □

Remark II.25. All the infinite sums are just finite integrals, for example

$$\sum_{a=1}^{\lfloor r\rfloor-1}\sum_{b=\lfloor R\rfloor+1}^{\infty} m_{a,b}^{(i)} = \int_0^{\frac{1}{\lfloor R\rfloor+1}}\int_{f_{a,r}(t)}^{\frac{1}{a}} \frac{\mathrm{d}v\,\mathrm{d}t}{(1+tv)^2}. \tag{II.26}$$

◇

Example II.27. In this example we compute the asymptotic frequency with which simultaneously $D_{n-2} > 2.9$ and $D_n > 3.6$; see Figure 5 and Table 2. Also compare with Table 1 where some of the upper bounds for this case are listed.

a_n	a_{n+1}	Case	asymptotic frequency
1	1	(vi_a)	0.047
1	2	(vi_a)	0.025
1	>2	(i)	0.106
2	1	(vi_c)	0.025
2	2	(vi_a)	0.013
2	3	(vi_a)	0.090
2	>3	(i)	0.044
>2	1	(iii)	0.097
>2	2	(iii)	0.050
>2	3	(iii)	0.034
>2	>3	(v)	0.115

TABLE 2. The probabilities that $D_{n-2} > 2.9$ and $D_n > 3.6$ in the various cases.

Summing over the cases yields that for almost all $x \in [0,1)\setminus\mathbb{Q}$ the asymptotic frequency with which simultaneously $D_{n-2} > 2.9$ and $D_n > 3.6$ is 0.64.

From this we may compute the conditional probability that M_{Tong} is the sharp bound. Given that $D_{n-2} > 2.9$ and $D_n > 3.6$ the conditional probability that M_{Tong} is the sharp bound is 0.31. ◇

5. Results for C_n.

In [**61**], Tong states the following result as theorem without a proof.

Let $t > 1$, $T > 1$ be two real numbers and

$$\begin{aligned} K =&\frac{1}{2}\Bigg(\frac{1}{t-1}+\frac{1}{T-1}+a_na_{n+1}t\,T \\ &+\sqrt{\left(\frac{1}{t-1}+\frac{1}{T-1}+a_na_{n+1}tT\right)^2-\frac{4}{(t-1)(T-1)}}\Bigg). \end{aligned}$$

Then

(1) $C_{n-2} < t, C_n < T$ imply $C_{n-1} > K$;

(2) $C_{n-2} > t, C_n > T$ imply $C_{n-1} < K$.

This statement is not correct; assume for instance that $C_{n-2} < 1.1$ and $C_n < 1.4$, and that $a_n = a_{n+1} = 1$. Part (1) of Tong's result then implies that $C_{n-1} > 11.94$. However, by definition $C_{n-1} \in (1,2)$, so this bound is clearly wrong.

In this section we give the correct result. The bounds in our theorems are sharp. We start with the case that both C_{n-2} and C_n are larger than given reals, this is related to the case where D_{n-2} and D_n are smaller than given numbers.

Theorem II.28. *Let $t, T \in (1,2)$ and put*

$$F' = \frac{a_{n+1}+1}{(a_n a_{n+1} + a_n + 1)t - 1}, \quad G' = \frac{a_n + 1}{(a_n a_{n+1} + a_{n+1} + 1)T - 1}$$

and $$L' = t + T + a_n a_{n+1} tT - 2.$$

Assume $C_{n-2} > t$ and $C_n > T$.

(1) *If $\frac{1}{t-1} - a_n \geq G'$ and $\frac{1}{T-1} - a_{n+1} < F'$, then*

$$C_{n-1} < \frac{T}{(a_{n+1}+1)(T-1)}.$$

(2) *If $\frac{1}{t-1} - a_n < G'$ and $\frac{1}{T-1} - a_{n+1} \geq F'$, then*

$$C_{n-1} < \frac{t}{(a_n+1)(t-1)}.$$

(3) *In all other cases*

$$C_{n-1} < 1 + \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{2(t-1)(T-1)}.$$

The bounds are sharp.

Proof. The proof follows from the fact that $C_n = 1 + \frac{1}{D_n}$ and Theorem II.20. If $C_{n-2} > t$, then $D_{n-2} = \frac{1}{C_{n-2}-1} < \frac{1}{t-1}$ and likewise if $C_n > T$, then $D_n < \frac{1}{T-1}$. Setting $r = \frac{1}{t-1}$ and $R = \frac{1}{T-1}$, it directly follows from (II.13) that $F = F'$ and $G = G'$.

Consider case (1). The condition $\frac{1}{t-1} - a_n \geq G'$ is equivalent to $r - a_n \geq G$ and $\frac{1}{a_n+1} \leq \frac{1}{T-1} - a_{n+1} < F'$ is equivalent to $\frac{1}{a_n+1} \leq R - a_{n+1} < F$ in part (1) of Theorem II.20. We find that

$$C_{n-1} < \frac{\frac{1}{T-1} - a_{n+1}}{a_{n+1}+1} + 1 = \frac{T}{(a_{n+1}+1)(T-1)}.$$

The proof of the second case is similar. For the third case we use Theorem II.4 for M_{Tong}.

$$
\begin{aligned}
& C_{n-1} < 1 + \frac{1}{M_{\mathrm{Tong}}} \\
= \quad & 1 + \frac{2}{t+T+a_n a_{n+1} tT - 2 + \sqrt{[t+T+a_n a_{n+1} tT - 2]^2 - 4(t-1)(T-1)}} \\
= \quad & 1 + \frac{2}{L' + \sqrt{L'^2 - 4(t-1)(T-1)}} \cdot \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{L' - \sqrt{L'^2 - 4(t-1)(T-1)}} \\
= \quad & 1 + \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{2(t-1)(T-1)}.
\end{aligned}
$$

□

Example II.29. Take $t = 1.1, T = 1.4$ and $a_n = a_{n+1} = 1$. We find that $F' = 0.870, G' = 0.625$ and $L' = 2.04$. Since $\frac{1}{T-1} - a_{n+1} = \frac{3}{2} > F'$ case (1) of Theorem II.28 does not apply. The second case does not apply either, since $\frac{1}{t-1} - a_n = 9 > G'$. So we are in case (3) and $C_{n-1} < 1.50$. ◇

We state the next theorem without a proof, since it is similar to that of Theorem II.28. The only difference is that the proof is based on Theorem II.22 instead of Theorem II.20.

Theorem II.30. *Let $t, T \in (1,2)$ and F', G' and L' be as defined in Theorem II.28. Assume $C_{n-2} < t$ and $C_n < T$.*

(1) *If $\dfrac{1}{t-1} - a_n \geq G'$ and $\dfrac{1}{T-1} - a_{n+1} < F'$, then*
$$C_{n-1} > 1 + \frac{F'}{a_{n+1}+1}.$$
(2) *If $G' \leq \dfrac{1}{t-1} - a_n$ and $\dfrac{1}{T-1} - a_{n+1} < F'$, then*
$$C_{n-1} > 1 + \frac{G'}{a_n+1}.$$
(3) *If $\dfrac{1}{t-1} - a_n < \dfrac{1}{a_{n+1}+1}$ and $\dfrac{1}{T-1} - a_{n+1} < \dfrac{1}{a_n+1}$, then*
$$C_{n-1} > 1 + \frac{1}{(a_n+1)(a_{n+1}+1)}.$$
(4) *In all other cases*
$$C_{n-1} > 1 + \frac{L' - \sqrt{L'^2 - 4(t-1)(T-1)}}{2(t-1)(T-1)}.$$

The bounds are sharp.

III
Approximation results for α-Rosen fractions

In this chapter we generalize Borel's classical approximation results for the regular continued fraction expansion to the α-Rosen fraction expansion, using a geometric method. We use α-Rosen fractions to give a Haas-Series-type result about all possible good approximations for the α for which the Legendre constant is larger than the Hurwitz constant.

1. Introduction

We recall Legendre's Theorem I.8 which states that all approximations with quality smaller than $\frac{1}{2}$ are found by the RCF-algorithm:

If $p, q \in \mathbb{Z}$, $q > 0$, and $\gcd(p, q) = 1$, then

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \quad \text{implies that} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix} \quad \text{for some } n \geq 0.$$

We call the best possible coefficient of $\frac{1}{q^2}$ in this theorem the Legendre constant. It is $\frac{1}{2}$ for RCF expansions. For the nearest integer continued fraction expansion (NICF) the Legendre constant is g^2, where g is the golden number; see [**21**].

Also recall Borel's Theorem I.10 which states that at least one in every three approximations has quality smaller than $\frac{1}{\sqrt{5}}$ and that this constant cannot be improved. Even more so, for every longer sequence of consecutive approximations, the guaranteed minimum of their qualities remains $\frac{1}{\sqrt{5}}$ and can only be improved if we exclude irrationals x for which there is an integer N such that $a_n = 1$ for all $n > N$. This leads in a natural way to the spectra by Markoff and Lagrange; see [**9**].

Because $\frac{1}{\sqrt{5}} < \frac{1}{2}$ the results of Legendre and Borel imply Hurwitz's Theorem I.11 that states that for every irrational number x there exist infinitely many pairs of integers p and q, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}} \frac{1}{q^2}$$

We call the best possible coefficient of $\frac{1}{q^2}$ in this inequality the Hurwitz constant. It is $\frac{1}{\sqrt{5}}$ for RCF-expansions .

As mentioned in the introduction, J.C. Tong [**59, 60**] generalized Borel's result for the nearest integer continued fraction expansion (NICF). He showed that for the NICF there exists a 'spectrum,' i.e., there exists a sequence of constants $(c_k)_{k\geq 1}$, monotonically decreasing to $1/\sqrt{5}$, such that for all irrational numbers x the minimum of any block of $k+2$ consecutive NICF-approximation coefficients is smaller than c_k.

Theorem III.1. (Tong) *For every irrational number x and all positive integers n and k one has*

$$\min\{\Theta_{n-1},\Theta_n,\dots,\Theta_{n+k}\} < \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}.$$

The constant $c_k = \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}$ is best possible.

In [**17**] Hartono and Kraaikamp showed how Tong's result follows by a geometrical method based on the natural extension of the NICF. The method will be discussed in Section 2. In [**30**] this method was extended to Rosen fractions, yielding the next theorem.

Theorem III.2. *Fix $q = 2p$, with $p \in \mathbb{N}, p \geq 2$ and let $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$. For every G_q-irrational number x and all positive n and k, one has*

$$\min\{\Theta_{n-1},\Theta_n,\dots,\Theta_{n+k(p-1)}\} < c_k,$$

with

$$c_k = \frac{-\tau_{k-1}}{1+(\lambda-1)\tau_{k-1}} \quad \text{and} \quad \tau_k = \left[\left(-1:2,(-1:1)^{p-2}\right)^k, -2:3\right].$$

The constant c_k is best possible.

A similar theorem was derived for the case that q is odd. In this chapter we derive Borel results for both even and odd α-Rosen fractions.

1.1. α-Rosen fractions. The α-Rosen fraction operator is defined in (I.43) as

$$T_\alpha(x) = \frac{\varepsilon}{x} - \lambda\left\lfloor \frac{\varepsilon}{\lambda x} + 1 - \alpha \right\rfloor \text{ if } x \neq 0 \text{ and } T_\alpha(0) := 0. \tag{III.3}$$

Repeatedly applying this operator to $x \in [(\alpha-1)\lambda, \alpha\lambda)$ yields the α-Rosen expansion of x. Put

$$d(x) = \left\lfloor \left|\frac{1}{\lambda x}\right| + 1 - \alpha \right\rfloor \quad \text{and} \quad \varepsilon(x) = \mathrm{sgn(x)}. \tag{III.4}$$

Furthermore, for $n \geq 1$ with $T_\alpha^{n-1}(x) \neq 0$ put

$$\varepsilon_n(x) = \varepsilon_n = \epsilon(T_\alpha^{n-1}(x)) \text{ and } d_n(x) = d_n = d(T_\alpha^{n-1}(x)).$$

This yields a continued fraction of the type

$$x = \cfrac{\varepsilon_1}{d_1\lambda + \cfrac{\varepsilon_2}{d_2\lambda + \dots}} = [\varepsilon_1 : d_1, \varepsilon_2 : d_2, \dots],$$

where $\varepsilon \in \{\pm 1\}$ and $d_i \in \mathbb{N}^+$.

In this chapter we derive a Borel-type result for α-Rosen fractions. Let q be fixed. As mentioned in the introduction, Haas and Series [**16**] showed that for every G_q-irrational x there exist infinitely many G_q-rationals r/s, such that $s^2 \left|x - \frac{r}{s}\right| \leq \mathcal{H}_q$, where the Hurwitz constant $\mathcal{H}_q$ is given by

$$\mathcal{H}_q = \begin{cases} \dfrac{1}{2} & \text{if } q \text{ is even,} \\ \dfrac{1}{\sqrt{\lambda^2 - 4\lambda + 8}} & \text{if } q \text{ is odd.} \end{cases} \tag{III.5}$$

Theorem III.6. *Let $\alpha \in [1/2, 1/\lambda]$ and denote the nth α-Rosen convergent by p_n/q_n. For every G_q-irrational x there are infinitely many $n \in \mathbb{N}$ for which*

$$q_n^2 \left| x - \frac{p_n}{q_n} \right| \leq \mathcal{H}_q$$

The constant $\mathcal{H}_q$ is best possible.

We remarked that for regular continued fractions the results of Borel and Legendre imply Hurwitz's result. For Rosen fractions, the case $\alpha = \frac{1}{2}$ it follows from Nakada [**42**] that the Legendre constant is smaller than the Hurwitz constant $\mathcal{H}_q$ (both in the odd and even case). This means that there might exist G_q-rationals r/s for which

$$\Theta(x, r/s) = s^2 \left| x - \frac{r}{s} \right| \leq \mathcal{H}_q,$$

i.e., with "quality" $\Theta(x, r/s)$ smaller than $\mathcal{H}_q$, that are not found as Rosen-convergents. So a direct continued fraction proof of the generalization by Haas and Series [**16**] of Hurwitz's results cannot be given for standard Rosen fractions.

1.2. Legendre and Lenstra constants. In the early 1980s H.W. Lenstra conjectured that for regular continued fractions for almost all x and all $z \in [0,1]$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n \mid \Theta_j(x) \leq z\}$$

exists and equals the distribution function F defined by

$$F(z) = \begin{cases} \dfrac{z}{\log 2} & \text{if } 0 \leq z \leq \frac{1}{2} \\ \\ \dfrac{1 - z + \log 2z}{\log 2} & \text{if } \frac{1}{2} \leq z \leq 1. \end{cases}$$

A version of this conjecture had been formulated by W. Doeblin [**12**] before. In 1983 W. Bosma *et al.* [**4**] proved the Doeblin-Lenstra-conjecture for regular continued fractions and Nakada's α-expansions for $\alpha \in \left[\frac{1}{2}, 1\right]$.

A prominent feature of F is that there exists a unique largest positive constant $\mathcal{L}$ such that $F(z)$ is linear for $z \in [0, \mathcal{L}]$. For the RCF we have $\mathcal{L} = \frac{1}{2}$. In [**42**], Nakada calls $\mathcal{L}$ the Lenstra constant and shows that for a large class of continued fractions this Lenstra constant is equal to the Legendre constant. Using Lenstra constants, it was shown in [**29**] that the so-called mediant Rosen map has a Legendre constant larger than the Hurwitz constant $\mathcal{H}_q$, thus yielding a Hurwitz result. These results were obtained using the Lenstra constant. We derive a Hurwitz result for each odd q, using certain α-Rosen fractions.

The outline of this chapter is as follows. In Section 2 we give some general definitions for the natural extensions for α-Rosen fractions and explain briefly how our method works. The even and odd case have different properties and we handle the details in two separate sections. The Borel result for the various subcases of even α-Rosen fractions are derived in Section 3, and the odd case is given in Section 4. In Section 5 we find the Lenstra constants $\mathcal{L}_\alpha$ for α-Rosen fractions and thereby conclude for which values of α we can derive a Hurwitz result.

2. The natural extension for α-Rosen fractions

In this section we introduce the necessary notation. Recall from (III.3) that

$$T_\alpha(t) = \frac{\varepsilon(t)}{t} - d(t)\lambda \quad \text{with} \quad \varepsilon(t) = \operatorname{sgn}(t) \quad \text{and} \quad d(t) = \left\lfloor \frac{\varepsilon}{\lambda t} + 1 - \alpha \right\rfloor.$$

We generalize Definition I.15 of the natural extension operator.

Definition III.7. For fixed q and α the natural extension map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ is given by

$$\mathcal{T}_\alpha(t, v) = \left(T_\alpha(t), \frac{1}{d(t)\lambda + \varepsilon(t)v} \right).$$

$\diamond$

The shape of the domain Ω_α on which the two-dimensional map $\mathcal{T}_\alpha$ is bijective a.e. was first described in [**11**]. We derive our results using a geometric method based on the natural extensions Ω_α. The shape of Ω_α depends on α and we give the explicit formulas for each of the different cases in the appropriate sections; see e.g. the beginning of Subsection 3.1 for Ω_α when q is even and $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$. The natural extension also depends on q, but for ease of notation we suppress this dependence and write Ω_α in stead of $\Omega_{\alpha,q}$.

We use the following notation in the remainder of this thesis

$$\Omega_\alpha^+ = \{(x, y) \in \Omega_\alpha \text{ with } x \geq 0\} \text{ and } \Omega_\alpha^- = \{(x, y) \in \Omega_\alpha \text{ with } x < 0\}$$

and also use these superscripts for subregions of Ω_α^+ or Ω_α^-.

We use constants l_n and r_n to describe Ω_α, where

$$\begin{aligned} l_0 &= (\alpha - 1)\lambda \quad && \text{and } l_n = T_\alpha^n(l_0), \quad && \text{for } n \geq 0, \\ r_0 &= \alpha\lambda && \text{and } r_n = T_\alpha^n(r_0), && \text{for } n \geq 0. \end{aligned}$$

The orbit of $-\frac{\lambda}{2}$ in the case $\alpha = \frac{1}{2}$ plays an important role in describing the natural extensions. We define $\varphi_j = T_{\frac{1}{2}}^j\left(-\frac{\lambda}{2}\right)$.

We set $\delta_d = \frac{1}{(\alpha+d)\lambda}$ for all $d \geq 1$. So if $\delta_d < x \leq \delta_{d-1}$, we have $d(x) = d$ and $\varepsilon(x) = +1$; also see (III.4). For x with $-\delta_{d-1} \leq x < -\delta_d$ we have $d(x) = d$ and $\varepsilon(x) = -1$.

We often use the auxiliary sequence B_n already given in (I.41) by

(III.8) $$B_0 = 0, \quad B_1 = 1, \quad B_n = \lambda B_{n-1} - B_{n-2}, \quad \text{for } n = 2, 3, \ldots.$$

Note that $B_n = \sin\frac{n\pi}{q} / \sin\frac{\pi}{q}$. If $q = 2p$ for $p \geq 2$, we find from $\sin\frac{(p-1)\pi}{2p} = \sin\frac{(p+1)\pi}{2p}$ that

(III.9) $$B_{p-1} = B_{p+1} = \frac{\lambda}{2} B_p \quad \text{and} \quad B_{p-2} = \left(\frac{\lambda^2}{2} - 1\right) B_p.$$

Similarly in the odd case with $q = 2h+3$ for $h \in \mathbb{N}$ we have that

(III.10) $$B_{h+1} = B_{h+2}, \quad B_h = (\lambda - 1) B_{h+1} \quad \text{and} \quad B_{h-1} = \left(\lambda^2 - \lambda - 1\right) B_{h+1}.$$

Similar to (I.13) we define for $x \in [l_0, r_0)$ with α-Rosen expansion $[\varepsilon_1 : d_1, \varepsilon_2 : d_2, \ldots]$ the future t_n and the past v_n of x at time $n \geq 1$ by

$$t_n = [\varepsilon_{n+1} : d_{n+1}, \varepsilon_{n+2} : d_{n+2}, \ldots] \quad \text{and} \quad v_n = [1 : d_n, \varepsilon_n : d_{n-1}, \ldots, \varepsilon_2 : d_1].$$

Again, we set $t_0 = x$ and $v_0 = 0$.

Remark III.11. For α-Rosen fractions it again holds that $\mathcal{T}_\alpha{}^n(x, 0) = (t_n, v_n)$ for $n \geq 0$. $\diamond$

The $(n-1)$st and nth approximation coefficients of x can be given in terms of t_n and v_n (see Section 5.1.2 of [**10**]) as

(III.12) $$\Theta_{n-1} = \Theta_{n-1}(t_n, v_n) = \frac{v_n}{1 + t_n v_n} \quad \text{and} \quad \Theta_n = \Theta_n(t_n, v_n) = \frac{\varepsilon_{n+1} t_n}{1 + t_n v_n}.$$

Often it is convenient to use $\Theta_m(t_{n+1}, v_{n+1}) = \Theta_{m+1}(t_n, v_n)$.

Lemma III.13. *The $(n+1)$st approximation coefficient of x can be expressed in terms of $d_{n+1}, \varepsilon_{n+1}$ and ε_{n+2} by*

(III.14) $$\Theta_{n+1} = \Theta_{n+1}(t_n, v_n) = \frac{\varepsilon_{n+2}(1 - \varepsilon_{n+1} d_{n+1} t_n \lambda)(\lambda d_{n+1} + \varepsilon_{n+1} v_n)}{1 + t_n v_n}.$$

Proof. First we use (III.12) to write

$$\Theta_{n+1} = \Theta_n(t_{n+1}, v_{n+1}) \frac{\varepsilon_{n+2}\, t_{n+1}}{1 + t_{n+1}\, v_{n+1}} = \varepsilon_{n+2}\, t_{n+1} \frac{\Theta_n}{v_{n+1}} = \frac{\varepsilon_{n+2}\, \varepsilon_{n+1} \frac{t_{n+1} t_n}{v_{n+1}}}{1 + t_n v_n}.$$

Then we use $t_{n+1} = \frac{\varepsilon_{n+1}}{t_n} - d_{n+1}\lambda$ and $v_{n+1} = \frac{1}{\lambda d_{n+1} + \varepsilon_{n+1} v_n}$ to find (III.14). $\square$

In view of (III.12) we define functions f and g on $[l_0, r_0)$ by

(III.15) $$f(x) = \frac{\mathcal{H}_q}{1 - \mathcal{H}_q x} \quad \text{and} \quad g(x) = \frac{|x| - \mathcal{H}_q}{\mathcal{H}_q x}.$$

Then for points $(t_n, v_n) \in \Omega_\alpha$ one has

(III.16)

$$\Theta_{n-1} \leq \mathcal{H}_q \Leftrightarrow v_n \leq f(t_n) \quad \text{and} \quad \Theta_n \leq \mathcal{H}_q \Leftrightarrow \begin{cases} v_n \leq g(t_n) & \text{if } t_n < 0 \\ \\ v_n \geq g(t_n) & \text{if } t_n \geq 0. \end{cases}$$

We define $\mathcal{D}$ as

$$\text{(III.17)} \qquad \mathcal{D} = \left\{ (t,v) \in \Omega_\alpha \,\middle|\, \min\left\{ \frac{v}{1+tv}, \frac{|t|}{1+tv} \right\} > \mathcal{H}_q \right\},$$

so $\min\{\Theta_{n-1}, \Theta_n\} > \mathcal{H}_q$ if and only if $(t_n, v_n) \in \mathcal{D}$.

See Figure 2 for an example of the position of $\mathcal{D}$ and of the graphs of f and g in Ω_α for $q = 4$.

3. Tong's spectrum for even α-Rosen fractions

Let $q = 2p$ for $p \in \mathbb{N}^+$, $p \geq 2$ and set $\lambda = 2\cos\frac{\pi}{q}$. As shown in [**11**] there are three subcases for the shape of Ω_α: we need to study $\alpha = \frac{1}{2}, \alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$ and $\alpha = \frac{1}{\lambda}$ separately. The following result, giving the ordering of the l_n and r_n, was essential in the construction of the natural extensions.

Theorem III.18. [**11**] *Let $q = 2p, p \in \mathbb{N}, p \geq 2$ and let l_n and r_n be defined as before. If $\frac{1}{2} < \alpha < \frac{1}{\lambda}$, then we have that*

$$-1 < l_0 < r_1 < l_1 < \ldots < r_{p-2} < l_{p-2} < -\delta_1 < r_{p-1} < 0 < l_{p-1} < r_0 < 1,$$

$d_p(r_0) = d_p(l_0) + 1$ and $l_p = r_p$. If $\alpha = \frac{1}{2}$, then we have that

$$-1 < l_0 < r_1 = l_1 < \ldots < r_{p-2} = l_{p-2} < -\delta_1 < r_{p-1} = 0 = l_{p-1} < r_0 < 1.$$

If $\alpha = \frac{1}{\lambda}$, then we have that

$$-1 < l_0 = r_1 < l_1 = r_2 < \ldots < l_{p-2} = -\delta_1 = r_{p-1} < 0 < r_0 = 1.$$

Let $k \geq 1$ be an integer and put

$$\text{(III.19)} \qquad (\tau_k, \nu_k) = \begin{cases} \mathcal{T}_\alpha^{-k(p-1)}\left(\frac{-2}{3\lambda}, \lambda - 1\right) & \text{if } \alpha = \frac{1}{\lambda}; \\[2ex] \mathcal{T}_\alpha^{-k(p-1)}\left(-\delta_1, \lambda - 1\right) & \text{otherwise.} \end{cases}$$

We prove the following result in this section.

Theorem III.20. *Fix an even $q = 2p$ with $p \geq 3$ and let $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$. There exists a positive integer K such that for every G_q-irrational number x and all positive n and $k > K$, one has*

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)}\} < c_k \quad \textit{with } c_k = \frac{-\tau_{k-1}}{1 + \tau_{k-1}\nu_{k-1}}.$$

For every integer $k \geq 1$ we have $c_{k+1} < c_k$. Furthermore $\lim_{k\to\infty} c_k = \dfrac{1}{2}$.

The case $\alpha = \frac{1}{2}$ was proven in [**30**]; c.f. Theorem III.2. The proof for $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$ follows the same line and is given in Section 3.1. In Section 3.2 we use that the natural extension for the case $\alpha = \frac{1}{\lambda}$ is the reflection of the one for $\alpha = \frac{1}{2}$ to prove Theorem III.20 for $\alpha = \frac{1}{\lambda}$.

3.1. Even case with $\alpha \in (\frac{1}{2}, \frac{1}{\lambda})$. In this section we assume that $\alpha \in (\frac{1}{2}, \frac{1}{\lambda})$. In [**11**] the shape of $\Omega_\alpha$was determined.

Definition III.21. Set

$$\begin{aligned} J_{2n-1} &= [l_{n-1}, r_n) \quad \text{and} \quad J_{2n} = [r_n, l_n) \quad \text{for} \quad n = 1, 2, \ldots, p-1 \\ J_{2p-1} &= [l_{p-1}, r_0) \quad \text{and} \\ H_1 &= \frac{1}{\lambda+1}, \quad H_2 = \frac{1}{\lambda} \quad \text{and} \\ H_n &= \frac{1}{\lambda - H_{n-2}} \quad \text{for} \quad n = 3, 4, \ldots, 2p-1. \end{aligned}$$

The domain of Ω_α upon which $\mathcal{T}_\alpha$ is bijective a.e. is given by

$$\Omega_\alpha = \bigcup_{n=1}^{2p-1} J_n \times [0, H_n].$$

◇

See Figure 1 for an example of the shape of Ω_α for $q = 6$ and $\alpha = 0.53$.

We define

$$\Omega_\alpha^+ = \{(t, v) \in \Omega_\alpha \mid t > 0\}.$$

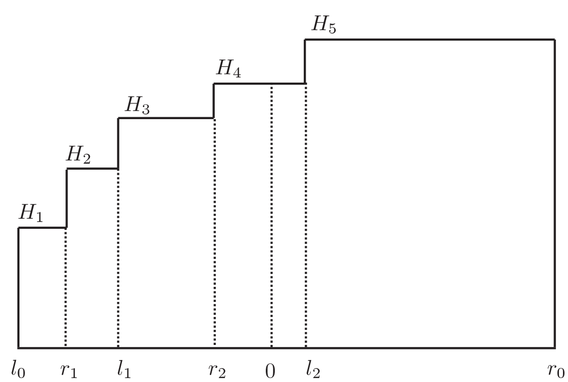


FIGURE 1. The natural extension Ω_α for $q = 6$ and $\alpha = 0.53 < \frac{1}{\lambda}$.

From the above description of the natural extension Ω_α it follows that the natural extension has $2p-1$ heights $H_1, \ldots, H_{2p-1}$. In [**11**] it is shown that $H_{i+1} > H_i$ for $i = 1, \ldots, 2p-2$,

$$H_{2p-3} = \lambda - 1, \quad H_{2p-2} = \frac{\lambda}{2}, \quad H_{2p-1} = 1,$$

and

(III.22)

$$l_{p-2} = \frac{\alpha\lambda^2 - 2}{(-\alpha\lambda^2 + 2\alpha + 1)\lambda}, l_{p-1} = \frac{(2\alpha-1)\lambda}{2 - \alpha\lambda^2} \text{ and } r_{p-1} = -\frac{(2\alpha-1)\lambda}{2 - (1-\alpha)\lambda^2}.$$

Note that Theorem III.20 gives a result for $q \geq 6$. The case $q = 4$ behaves essentially in the same way, which we show in the following section.

3.1.1. *The case* $q = 4$. In this subsection we assume that $q = 2p = 4$, so $\lambda = \sqrt{2}$. In this case we have $H_1 = \sqrt{2} - 1, H_2 = \frac{1}{2}\sqrt{2}$ and $H_3 = 1$. We prove the following result.

Theorem III.23. *Let* $\lambda = 2\cos\frac{\pi}{4} = \sqrt{2}$, *let* $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$ *and let* τ_k *be as given in (III.19). There exists a positive integer* K *such that for every* G_q*-irrational number* x *and all positive* n *and* $k \geq K$, *one has*

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k}\} < c_k \quad \text{with } c_k = \frac{\sqrt{2}-1}{1+\tau_{k-1}(\sqrt{2}-1)}.$$

For every integer $k \geq 1$ *we have* $c_{k+1} < c_k$. *Furthermore* $\lim_{k\to\infty} c_k = \dfrac{1}{2}$.

We start by determining the shape of the region $\mathcal{D} \subset \Omega_\alpha$, where $\min\{\Theta_{n-1}, \Theta_n\} > \frac{1}{2}$; see (III.17).

Lemma III.24. *Put* $\alpha_0 := \frac{4+\sqrt{2}}{8} = 0.676\ldots$. *For* $\alpha \in \left(\frac{1}{2}, \alpha_0\right)$ *the region* $\mathcal{D}$ *consists of one component* $\mathcal{D}_1$, *which is bounded by the lines* $t = l_0$, $v = H_1$, *and the graph of* f; *see Figure 2. If* $\alpha \in \left[\alpha_0, \frac{\sqrt{2}}{2}\right)$, *then* $\mathcal{D}$ *consists of two components:* $\mathcal{D}_1$ *and* $\mathcal{D}_2$, *where* $\mathcal{D}_2$ *is the region bounded by the lines* $t = r_1$, $v = H_2$, *and the graph of* g; *see Figure 3.*

Proof. Recall that $\mathcal{H}_q = \frac{1}{2}$. First assume that $t \geq 0$. The graphs of f and g do not intersect for $t \leq r_0$. Thus every point $(t_n, v_n) \in \Omega^+_\alpha$ is below the graph of f or above the graph of g. By (III.16) we have when $t \geq 0$ that $\min\{\Theta_{n-1}, \Theta_n\} < \frac{1}{2}$.

Assume that $t < 0$. The graphs of f and g intersect with the line $v = H_1$ in the point $(1-\sqrt{2}, \sqrt{2}-1)$, and we find that $l_0 < -\frac{1}{2} < -\delta_1 < 1-\sqrt{2} < r_1$ for $\alpha \in (\frac{1}{2}, \frac{1}{\lambda})$. Since both f and g are monotonically increasing we find that for $l_0 < t < r_1$ the intersection of $\mathcal{D}$ and Ω_α is $\mathcal{D}_1$.

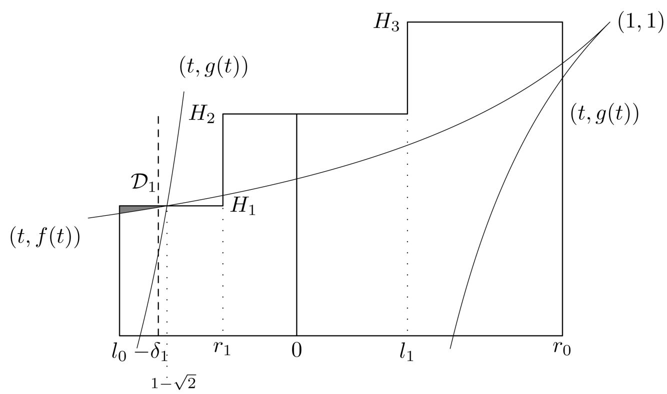


Figure 2. Here $q = 4$ and $\alpha = 0.6 < \alpha_0 = 0.676\ldots$. The region $\mathcal{D}$ consists of one component $\mathcal{D}_1$.

One easily checks that $g(r_1) \leq H_2$ if and only if $\alpha \geq \alpha_0$. So we find that for $\alpha > \alpha_0$ and $r_1 < t < 0$ the intersection of $\mathcal{D}$ and Ω_α is $\mathcal{D}_2$ (for $\alpha = \alpha_0$ the region $\mathcal{D}_2$ consists of exactly one point, $(r_1, g(r_1))$); see Figure 3. $\square$

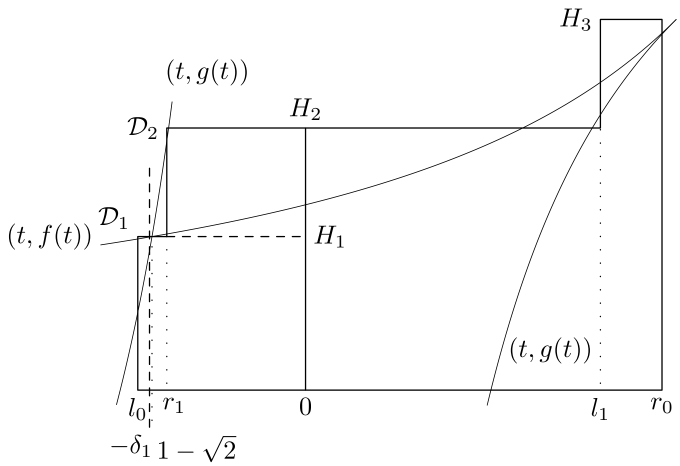


FIGURE 3. Here $q = 4$ and $\alpha = 0.68 > \alpha_0 = 0.676\ldots$. The region $\mathcal{D}$ consists of two components.

Proof of Theorem III.23. Recall that $(\tau_k, \nu_k) = \mathcal{T}_\alpha^{-k}(-\delta_1, \lambda - 1)$. For $q = 4$ we find $\tau_1 = \frac{-1}{2\sqrt{2}-\delta_1}$, $\tau_k = [(-1 : 2)^k, -\delta_1]$ and $\nu_k = \sqrt{2} - 1$ for all k. Since T_α is strictly increasing on the interval $[-\delta_1, -\delta_2)$ we find $\tau_{k-1} < \tau_k$. In this case $\lim_{k\to\infty} \tau_k = [\overline{(-1 : 2)}] = \dfrac{-1}{\sqrt{2}+1} = 1 - \sqrt{2}$. We conclude that $\lim_{k\to\infty} \tau_k = 1 - \sqrt{2}$.

We find $c_{k+1} < c_k$, $\lim_{k\to\infty} c_k = \dfrac{\sqrt{2}-1}{1 + (1-\sqrt{2})(\sqrt{2}-1)} = \dfrac{1}{2}$ and conclude that $c_k > \frac{1}{2}$ for all k.

We now focus on the orbit of points in $\mathcal{D}$ and start with $\mathcal{D}_1$. First note that $\mathcal{T}_\alpha([l_0, -\delta_1) \times [0, H_1]) = [l_1, r_0) \times [H_2, 1] \subset \{(t, v) \in \Omega_\alpha | \, t \geq 0\}$. Therefore, if $(t_n, v_n) \in \mathcal{D}_1$ and $t_n \leq -\delta_1$, we have that $\min\{\Theta_{n-1}, \Theta_n\} > \frac{1}{2}$, while $\Theta_{n+1} < \frac{1}{2}$. For these points we have proven Theorem III.23 with $K = 1$.

Note that $(1 - \sqrt{2}, \sqrt{2} - 1)$ is a fixed-point of $\mathcal{T}_\alpha$. In particular, we have that $(1-\sqrt{2}, \sqrt{2}-1)$ is a repellent fixed-point for the first-coordinate map of $\mathcal{T}_\alpha$, and an attractive fixed point for the second coordinate map of $\mathcal{T}_\alpha$. Thus points $(t, v) \in \mathcal{D}_1$ with $t > -\delta_1$ move "left and up" under $\mathcal{T}_\alpha$. Noting that f is strictly increasing, we have that the region $\{(t, v) \in \mathcal{D}_1 | \, t \geq -\delta_1\}$ is mapped by $\mathcal{T}_\alpha$ inside $\mathcal{D}_1$.

Setting $\mathcal{D}_{1,k} := \{(t, v) \in \mathcal{D}_1 | \, \tau_{k-1} \leq t < \tau_k\}$, for $k \geq 1$, and $\mathcal{D}_{1,0} := \{(t, v) \in \mathcal{D}_1 | \, t < -\delta_1\}$, by definition of τ_k and $\mathcal{D}_{1,k}$, we have for $k \geq 1$ that

$$(t, v) \in \mathcal{D}_{1,k} \quad \text{implies} \quad \mathcal{T}_\alpha(t, v) \in \mathcal{D}_{1,k-1}.$$

We determine the maximum of Θ_{n-1}, Θ_n and Θ_{n+1} on $\mathcal{D}_{1,k}$ for $k \geq 1$.

Lemma III.25. *Let $k \geq 1$ and $(t_n, v_n) \in \mathcal{D}_{1,k}$. Then*

$$\Theta_{n-1} \leq \Theta_n \leq \Theta_{n+1}. \tag{III.26}$$

PROOF. On $\mathcal{D}_{1,k}$ we have $d_{n+1} = 2$ and $\varepsilon_{n+1} = \varepsilon_{n+2} = -1$. From (III.12) we find

$$\Theta_{n-1} \leq \Theta_n \quad \text{if and only if} \quad v_n \leq -t_n,$$

and the latter inequality is true in view of the fixed point. From (III.14) we have $\Theta_{n+1} = \frac{-(1+2\sqrt{2}t_n)(2\sqrt{2}-v_n)}{1+t_n v_n}$ and we find

$$\Theta_n \leq \Theta_{n+1} \quad \text{if and only if} \quad v_n \leq \frac{7t_n + 2\sqrt{2}}{1+2\sqrt{2}t_n}.$$

On $\mathcal{D}_{1,k}$ the function $v(t) = \frac{7t+2\sqrt{2}}{1+2\sqrt{2}t}$ is decreasing in t and $v(1-\sqrt{2}) = \sqrt{2}-1$. So $\Theta_n \leq \Theta_{n+1}$ on $\mathcal{D}_{1,k}$ for $k \geq 1$. □

We conclude that for every point $(t_n, v_n) \in \mathcal{D}_{1,k}$ for $k \geq 1$

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} \leq \max_{(t,v)\in\mathcal{D}_{1,k}} \Theta_{n-1}(t,v).$$

We determine the maximum of Θ_{n-1} on $\mathcal{D}_{1,k}$. The partial derivatives of Θ_{n-1} are given by

$$\frac{\partial\Theta_{n-1}}{\partial t_n} = \frac{-v_n^2}{(1+t_n v_n)^2} < 0 \quad \text{and} \quad \frac{\partial\Theta_{n-1}}{\partial v_n} = \frac{1}{(1+t_n v_n)^2} > 0,$$

so on $\mathcal{D}_{1,1}$ we see that Θ_{n-1} attains its maximum in $(-\delta_1, \sqrt{2}-1)$. Similarly on $\mathcal{D}_{1,k}$ we find that Θ_{n-1} attains its maximum in $(\tau_k, \sqrt{2}-1)$, which is the top left-hand vertex of $\mathcal{D}_{1,k}$. We find the values

$$c_1 = \frac{\sqrt{2}-1}{1-\delta_1(\sqrt{2}-1)} \quad \text{and} \quad c_k = \frac{\sqrt{2}-1}{1+\tau_{k-1}(\sqrt{2}-1)}.$$

One sees that if $(t_n, v_n) \in \mathcal{D}_{1,k}$ for some $k \geq 3$, then $(t_{n+1}, v_{n+1}) \in \mathcal{D}_{1,k-1}$, $(t_{n+2}, v_{n+2}) \in \mathcal{D}_{1,k-2}, \ldots, (t_{n+k-1}, v_{n+k-1}) \in \mathcal{D}_{1,1}$. It follows that

$$(t_n, v_n) \in \mathcal{D}_{1,k} \quad \text{implies} \quad \frac{1}{2} < \min\{\Theta_{n-1}, \ldots, \Theta_{n+k}\} < \Theta_{n-1}(\tau_{k-1}, \sqrt{2}-1) = c_k$$
$$\text{and } \Theta_{n+k+1} < \frac{1}{2}. \tag{III.27}$$

For $\alpha < \alpha_0$ the above implication (III.27) is actually an equivalence, since $\mathcal{D}_2$ is void for these values of α. Thus Theorem III.23 for the case $q = 4$ and $\alpha < \alpha_0$ follows with $K = 1$.

We continue by studying the orbit of points in $\mathcal{D}_2$ and assume that $\alpha \in \left[\alpha_0, \frac{1}{\lambda}\right)$, so $\mathcal{D}_2$ is non-empty.

It follows from the fact that $(1-\sqrt{2}, \sqrt{2}-1)$ is a repellent fixed-point on the first coordinate map of $\mathcal{T}_\alpha$, and an attractive fixed-point on the second coordinate map of $\mathcal{T}_\alpha$, that for $(t,v) \in \mathcal{D}_2$

$$1-\sqrt{2} < r_1 < t < \pi_1(\mathcal{T}_\alpha(t,v)) \quad \text{and} \quad H_2 > v > \pi_2(\mathcal{T}_\alpha(t,v)) > \sqrt{2}-1,$$

(here π_i is the projection on the ith coordinate), i.e., $\mathcal{T}_\alpha$ "moves" the point $(t,v) \in \mathcal{D}_2$ to the right, and "downwards towards" $\sqrt{2}-1$.

Let $(t_1, \sqrt{2}-1+v_1)$ be a point in $\mathcal{D}_2$. The lowest point in $\overline{\mathcal{D}_2}$, the closure of $\mathcal{D}_2$, is given by $(r_1, g(r_1)) = \left(\frac{1-2\alpha}{\sqrt{2}\alpha}, \frac{(\sqrt{2}-4)\alpha+2}{2\alpha-1}\right)$, so $0 < \frac{-(2+\sqrt{2})\alpha+1+\sqrt{2}}{2\alpha-1} < v_1$ for every point $(t_1, \sqrt{2}-1+v_1) \in \mathcal{D}_2$ and trivially $v_1 < 1$.

For the second coordinate we find

$$\pi_2(\mathcal{T}_\alpha(t_1, \sqrt{2}-1+v_1)) = \frac{1}{\sqrt{2}+1-v_1} = \sqrt{2}-1+\frac{\sqrt{2}-1}{\sqrt{2}+1-v_1}v_1.$$

For all points in $\mathcal{D}_2$ we have $d = 2$ and $\varepsilon = -1$. Thus in every consecutive step the second coordinate will be closer to the value $\sqrt{2}-1$ by a factor $\frac{\sqrt{2}-1}{\sqrt{2}+1-v_1} < \frac{\sqrt{2}-1}{\sqrt{2}} < 1$. Hence there exists a smallest positive integer K such that for all $(t, v) \in \mathcal{D}_2$ one has $\mathcal{T}_\alpha^K(t, v) \notin \mathcal{D}_2$. In words: the region $\mathcal{D}_2$ is "flushed" out of $\mathcal{D}$ in K steps, and the implication in (III.27) is an equivalence for $k > K$. This proves Theorem III.23. □

Remark III.28. More can be said with (considerable) effort. We start by deriving α_1 such that for $\alpha \in [\alpha_0, \alpha_1]$ the region $\mathcal{D}_2$ is non-empty, but flushed after one iteration of $\mathcal{T}_\alpha$. We find α_1 by solving for which value of α we have that the point $\mathcal{T}_\alpha(r_1, H_2) = \left(r_2, \frac{\sqrt{2}}{3}\right)$ is on the graph of g. Using that $d(r_1) = 2$ for $\alpha \in \left[\alpha_0, \frac{1}{\lambda}\right)$ we find that the only solution is given by $\alpha_1 := \frac{24+\sqrt{2}}{36} = 0.70595\ldots$.

So if $\alpha \in [\alpha_0, \alpha_1] = \left[\frac{4+\sqrt{2}}{8}, \frac{24+\sqrt{2}}{36}\right]$ and $(t_n, v_n) \in \mathcal{D}_2$, then $\min\{\Theta_{n-1}, \Theta_n\} > \frac{1}{2}$, and $\Theta_{n+1} < \frac{1}{2}$.

For $\alpha > \alpha_1$ and $i \geq 1$, we define the pre-images g_i of g for $t \in [1-\sqrt{2}, \beta]$ by

$$v = g_i(t) \quad \Leftrightarrow \quad \pi_2(\mathcal{T}_\alpha^i(t, v)) = g\big(\pi_1(\mathcal{T}_\alpha^i(t, v))\big),$$

i.e., the point (t, v) is on the graph of g_i if and only if $\mathcal{T}_\alpha^i(t, v)$ is on the graph of g. Note that for every $i \geq 1$ one has that $(1-\sqrt{2}, \sqrt{2}-1)$ is on the graph of g_i.

By definition of K_α it follows that the graph of g_i has a non-empty intersection with $\mathcal{D}_2$ if and only if $i = 1, \ldots, K_\alpha - 1$. These $K_\alpha - 1$ graphs g_i divide $\mathcal{D}_2$ like a "cookie-cutter" into regions $\mathcal{D}_{2,i}$ for $i = 1, \ldots, K_\alpha$; setting $g_0 := g$,

$$\mathcal{D}_{2,i} := \{(t, v) \in \mathcal{D}_2 |\, g_{i-1}(t) \leq v < \min\{H_2, g_i(t)\}\}.$$

We have that

$$(t_n, v_n) \in \mathcal{D}_{2,i} \quad \Rightarrow \quad \min\{\Theta_{n-1}, \ldots, \Theta_{n+i-1}\} > \frac{1}{2} \text{ and } \Theta_{n+i} < \frac{1}{2}.$$

In principle it is possible to determine the optimal constant $\tilde{c}_k$ for $(t_n, v_n) \in \mathcal{D}_2$ and $k = 1, \ldots, K-1$ such that

$$\min\{\Theta_{n-1}, \ldots, \Theta_{n+k}\} < \tilde{c}_k.$$

In this way, Theorem III.20 can be further sharpened. ◇

Remark III.29. From the proof of Lemma III.26 it easily follows that for points $(t_n, v_n) \in \mathcal{D}_2$

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} = \Theta_n.$$

The partial derivatives of Θ_n on $\mathcal{D}_2$ are given by

$$\frac{\partial \Theta_n}{\partial t_n} = \frac{-1}{(1+t_n v_n)^2} < 0 \quad \text{and} \quad \frac{\partial \Theta_n}{\partial v_n} = \frac{t_n^2}{(1+t_n v_n)^2} > 0.$$

$\diamond$

Example III.30. An easy but tedious calculation yields that $\mathcal{T}_\alpha^2(r_1, H_2)$ is on the graph of g if $\alpha = \alpha_2 := \frac{140+\sqrt{2}}{200} = 0.707071\ldots$.

For $\alpha \in (\alpha_1, \alpha_2]$ we have that $K_\alpha = 2$. For $\alpha \in (\alpha_1, \alpha_2]$ the region $\mathcal{D}_2$ consists of two parts: $\mathcal{D}_{2,1}$ and $\mathcal{D}_{2,2}$. The region $\mathcal{D}_{2,1}$ is immediately flushed and is therefore not interesting for us. For $(t_n, v_n) \in \mathcal{D}_{2,2}$ we have that

$$\frac{1}{2} < \min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \Theta_n\left(\mathcal{T}_\alpha(r_1, H_2)\right) = 9\sqrt{2}\alpha - 6\sqrt{2}.$$

The comparable value of c_1 is

$$c_1 = \frac{\delta_1}{1-\delta_1 H_1} = \frac{(\alpha+1)(2-\sqrt{2})}{\sqrt{2}\alpha + 2\sqrt{2} - 1}.$$

We find that $9\sqrt{2}\alpha - 6\sqrt{2} > \frac{(\alpha+1)(2-\sqrt{2})}{\sqrt{2}\alpha+2\sqrt{2}-1}$ when $\alpha > \frac{12-9\sqrt{2}+\sqrt{378+216\sqrt{2}}}{36} = 0.6944$. So we find for $\alpha \in (\alpha_1, \alpha_2]$ that

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < 9\sqrt{2}\alpha - 6\sqrt{2}.$$

$\diamond$

3.1.2. *Even case with $\alpha \in (\frac{1}{2}, \frac{1}{\lambda})$ and $q \geq 6$.* For the remainder of this section we assume $q \geq 6$. The shape of the region $\mathcal{D} \subset \Omega_\alpha$, where $\min\{\Theta_{n-1}, \Theta_n\} > \frac{1}{2}$ is given in the next lemma; see Figure 4.

Lemma III.31. *For $\alpha \in (\frac{1}{2}, \frac{\lambda^2+4\lambda-4}{2\lambda^3}]$ the region $\mathcal{D}$ consists of two components $\mathcal{D}_1$ and $\mathcal{D}_2$. The subregion $\mathcal{D}_1$ is bounded by the lines $t = l_0, v = H_1$ and the graph of f; $\mathcal{D}_2$ is bounded by the graph of g from the right, by the graph of f from below and by the boundary of Ω_α.*

If $\alpha \in \left(\frac{\lambda^2+4\lambda-4}{2\lambda^3}, \frac{-\lambda^2+4\lambda+4}{8\lambda}\right]$, then $\mathcal{D}_2$ splits into two parts and $\mathcal{D}$ consists of three components.

If $\alpha \in \left(\frac{-\lambda^2+4\lambda+4}{8\lambda}, \frac{1}{\lambda}\right)$, then $\mathcal{D}$ consists of four components: $\mathcal{D}_1$, the two parts of $\mathcal{D}_2$ and an additional part $\mathcal{D}_3$, bounded by the line $t = r_{p-1}$, the graph of g and the line $v = H_{2p-2}$; see Figure 5.

Proof. Recall that $\mathcal{H}_q = \frac{1}{2}$. First assume $t \geq 0$. Arguing as in the case of $q = 4$, when $t \geq 0$ we have $\min\{\Theta_{n-1}, \Theta_n\} < \frac{1}{2}$.

Assume $t < 0$. The graph of $f(t)$ intersects the line $v = H_1$ in the point $(-H_{2p-3}, H_1)$, but also $l_0 < -H_{2p-3} < r_1$ if $\alpha < \frac{1}{\lambda}$. Since the function $f(t)$ is strictly increasing and $f(0) = \mathcal{H}_q < \frac{1}{\lambda} = H_2$ it follows that the graph of $f(t)$ does not intersect any of the line segments $v = H_i$ for $i = 2, \ldots, 2p-2$ for $t < 0$.

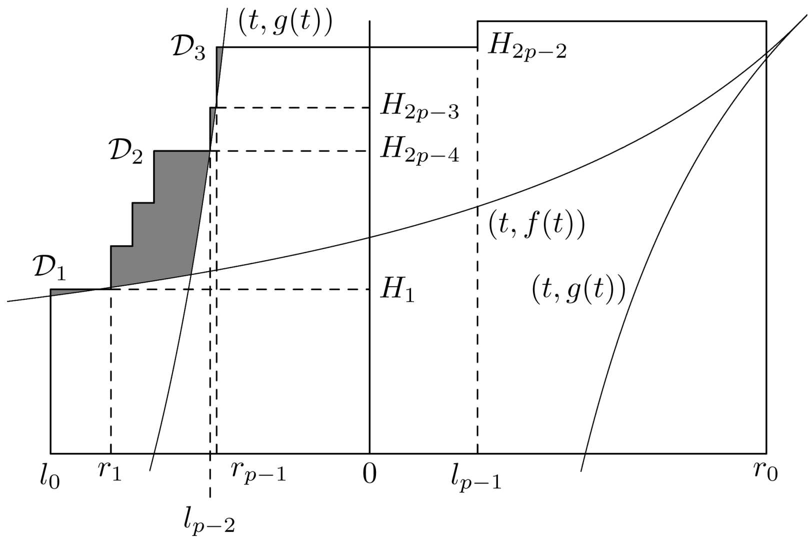


FIGURE 4. Sketch of $\mathcal{D}$ in Ω_α. The number of steps on the left boundary of $\mathcal{D}_2$ is about $p-4$. In this figure we took $\alpha \in (\frac{-\lambda^2+4\lambda+4}{8\lambda}, \frac{1}{\lambda})$, so $\mathcal{D}_2$ is split into two components and there is a region $\mathcal{D}_3$.

Next we consider the intersection points of the graph of $g(t)$ with the line segments $v = H_i$ for $i = 1, \ldots, 2p-2$. We work from right to left. The intersection point of the graph of g and the line $v = H_{2p-2} = \frac{\lambda}{2}$ is given by $\left(\frac{-2}{\lambda+4}, \frac{\lambda}{2}\right)$. The first coordinate of this point is larger than r_{p-1} if and only if $\alpha > \frac{-\lambda^2+4\lambda+4}{8\lambda}$ and always smaller than l_{p-1}, since $l_{p-1} > 0$. We conclude that $\frac{-2}{\lambda+4}$ is in the interval J_{2p-2} if and only if $\alpha \in \left(\frac{-\lambda^2+4\lambda+4}{8\lambda}, \frac{1}{\lambda}\right)$.

The intersection point of the graph of $g(t)$ with the line $v = H_{2p-3} = \lambda - 1$ is given by $(-H_1, H_{2p-3})$. Since $-\delta_1 < -H_1 < r_{p-1}$ we have by Theorem III.18 that $-H_1 \in J_{2p-3}$ for all $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$.

Furthermore $g(l_{p-2}) = -2 - \frac{(-\alpha\lambda^2+2\alpha+1)\lambda}{\alpha\lambda^2-2}$ and we find that $g(l_{p-2}) > H_{2p-4} = \lambda - \frac{2}{\lambda}$ if and only if $\alpha > \frac{\lambda^2+4\lambda-4}{2\lambda^3}$. So if $\alpha \in \left(\frac{\lambda^2+4\lambda-4}{2\lambda^3}, \frac{1}{\lambda}\right)$ then $\mathcal{D}_2$ consists of two separated parts.

The graph of $g(t)$ does not intersect any of the other lines $v = H_i$ with $i = 1, \ldots, 2p-5$, since g is strictly increasing and $g(r_{p-2}) = -2 - \frac{1}{r_{p-2}} = -2 + \lambda + r_{p-1} < 0$, where we used that $r_{p-2} = \frac{-1}{\lambda + r_{p-1}}$, $\lambda < 2$ and $r_{p-1} < 0$. $\square$

We see that $\mathcal{D}$ stretches over several intervals J_n - which was not the case for $q = 4$. Points (t_n, v_n) in $\mathcal{D}_1$ have $t_n \in J_1 = [l_0, r_1)$, points in $\mathcal{D}_2$ have $t_n \in J_2 \cup \cdots \cup J_{2p-1} = [r_1, r_{p-1})$ and points in $\mathcal{D}_3$ have $t_n \in J_{2p-2} = [r_{p-1}, l_{p-1}]$.

On $\mathcal{D}$ we consider Θ_{n+1}, the "next" approximation coefficient. We wish to express Θ_{n+1} locally as a function of only t_n and v_n. We divide $\mathcal{D}$ into subregions where $d_{n+1}, \varepsilon_{n+1}$ and ε_{n+2} are constant. This gives three regions; see Table 1 for the definition of the subregions.

	Region	d_{n+1}	ε_{n+1}	ε_{n+2}
(I)	$\left\{(t_n, v_n) \in \mathcal{D} \mid l_0 \leq t_n < \frac{-1}{\lambda}\right\}$	1	-1	-1
(II)	$\left\{(t_n, v_n) \in \mathcal{D} \mid \frac{-1}{\lambda} \leq t_n < -\delta_1\right\}$	1	-1	1
(III)	$\left\{(t_n, v_n) \in \mathcal{D} \mid -\delta_1 \leq t_n < \frac{-1}{2\lambda}\right\}$	2	-1	-1

Table 1. Subregions of $\mathcal{D}$ giving constant coefficients.

We analyse Θ_{n+1} on the three regions.

Region(I). On Region (I) we have that $\Theta_{n+1} < \frac{1}{2}$ if and only if $v_n > \frac{2\lambda t_n + 2\lambda + 1}{2\lambda t_n - t_n + 2}$.

Region(II). Region (II) is mapped to Ω_α^+ under $\mathcal{T}_\alpha$, so on Region (II) $\Theta_{n+1} < \frac{1}{2}$ for all points (t_n, v_n).

Region(III). We denote the intersection of $\mathcal{D}_2$ and Region (III) by $\mathcal{A}$. The vertices of $\mathcal{A}$ are given by $(-\delta_1, g(-\delta_1))$, $\left(\frac{-1}{\lambda+1}, \lambda - 1\right)$ and $(-\delta_1, \lambda - 1)$. The vertices of $\mathcal{D}_3$ are given by $(r_{p-1}, g(r_{p-1}))$, $\left(\frac{-2}{\lambda+4}, \frac{\lambda}{2}\right)$ and $(r_{p-1}, \frac{\lambda}{2})$. See Figure 5.

We focus on Region (III) and discuss points in Region (I) later.

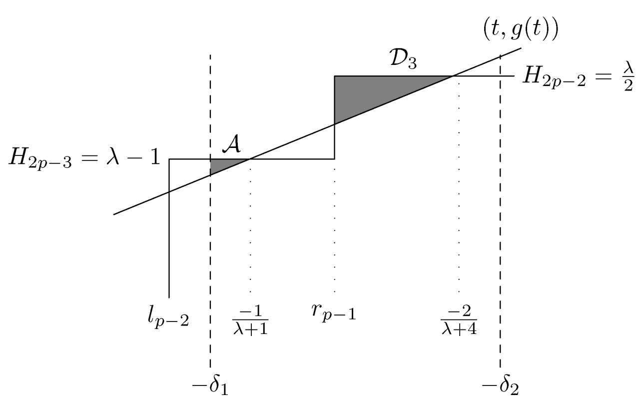


Figure 5. Region (III) in Ω_α. If $\alpha < \frac{-\lambda^2+4\lambda+4}{8\lambda}$ there is no region $\mathcal{D}_3$.

We want to determine bounds for the minimum of three consecutive approximation coefficients on $\mathcal{A}$ and $\mathcal{D}$.

Lemma III.32. *For each point* (t_n, v_n) *in Region* (III)

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} = \Theta_n.$$

Proof. On region (III) it holds that $v_n > -t_n$, so from (III.12) it immediately follows that

$$\Theta_{n-1} > \Theta_n.$$

Using (III.14) we find that $\Theta_{n+1} > \Theta_n$ if and only if $v_n < \frac{(4\lambda^2-1)t_n+2\lambda}{2\lambda t_n+1}$. Put $v(t) = \frac{(4\lambda^2-1)t+2\lambda}{2\lambda t+1}$. This function is decreasing in t for $t < \frac{-1}{2\lambda}$. We find that $v\left(\frac{-2}{\lambda+4}\right) > 1$ if and only if $\lambda > \frac{11+\sqrt{73}}{12}$. This last inequality is satisfied for all λ_q with $q \geq 6$. $\square$

Corollary III.33. *For each point* (t_n, v_n) *in Region* (III)

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n-1}\} > \frac{1}{2}.$$

It follows that for every point (t_n, v_n) in Region (III) we have

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} \leq \max_{(t,v)\in \text{ Region (III)}} \Theta_n(t,v).$$

The partial derivatives of Θ_n on Region (III) are given by

$$\frac{\partial \Theta_n}{\partial t_n} = \frac{-1}{(1+t_n v_n)^2} < 0 \quad \text{and} \quad \frac{\partial \Theta_n}{\partial v_n} = \frac{t_n^2}{(1+t_n v_n)^2} > 0.$$

We find that Θ_n takes its maximum on $\mathcal{A}$ in the upper left corner, the point $(-\delta_1, H_{2p-3})$, and on $\mathcal{D}_3$ in the vertex (r_{p-1}, H_{2p-2}). Using (III.22) we find that these maxima are given by

$$\begin{aligned}\Theta_n(-\delta_1, \lambda-1) &= \frac{\delta_1}{1-\delta_1(\lambda-1)} = \frac{1}{\alpha\lambda+1},\\ \Theta_n\left(r_{p-1}, \frac{\lambda}{2}\right) &= \frac{-r_{p-1}}{1+\frac{r_{p-1}\lambda}{2}} = \frac{2\lambda(2\alpha-1)}{4-\lambda^2}.\end{aligned}$$

We find that $\Theta_n(-\delta_1, \lambda-1) > \Theta_n\left(r_{p-1}, \frac{\lambda}{2}\right)$ if and only if $\alpha < \frac{\lambda-2+\sqrt{-3\lambda^2+4\lambda+20}}{4\lambda}$. For all $\lambda < 2$ we have $\frac{-\lambda^2+4\lambda+4}{8\lambda} < \frac{\lambda-2+\sqrt{-3\lambda^2+4\lambda+20}}{4\lambda} < \frac{1}{\lambda}$

Corollary III.34. *For every point* (t_n, v_n) *in Region (III)*

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} \leq \begin{cases} \frac{1}{\alpha\lambda+1} & \text{if } \alpha \in \left(\frac{1}{2}, \frac{\lambda-2+\sqrt{-3\lambda^2+4\lambda+20}}{4\lambda}\right], \\ \frac{2\lambda(2\alpha-1)}{4-\lambda^2} & \text{if } \alpha \in \left(\frac{\lambda-2+\sqrt{-3\lambda^2+4\lambda+20}}{4\lambda}, \frac{1}{\lambda}\right). \end{cases}$$

Orbit of points in Region (III). We study the orbit of points in $\mathcal{A}$ and $\mathcal{D}_3$ to derive the spectrum for α-Rosen fractions. We call $p-1$ consecutive applications of $\mathcal{T}_\alpha$ a *round*. We use Möbius transformations from Definition I.38 to find an explicit formula for $\mathcal{T}_\alpha^{p-1}$; see [**14**] for background, or [**30**], where such techniques are used. Let S and T be the generating matrices of the group G_q

$$S = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{III.35}$$

Recall that in this context we consider matrices M and $-M$ to be equivalent.

Lemma III.36. *Let $(t,v) \in \Omega_\alpha$ be given. Put $d = d(t)$, $\varepsilon = \varepsilon(t)$ and* $A = \begin{bmatrix} -d\lambda & \varepsilon \\ 1 & 0 \end{bmatrix}$. *Then*

$$\mathcal{T}_\alpha(t,v) = (A(t), TAT(v)).$$

Proof. Formula (III.3) gives

$$T_\alpha(t) = \frac{\varepsilon}{t} - d\lambda = \frac{-d\lambda t + \varepsilon}{t} = \begin{bmatrix} -d\lambda & \varepsilon \\ 1 & 0 \end{bmatrix} (t).$$

Now it easily follows that

$$TAT(v) = \begin{bmatrix} 0 & 1 \\ \varepsilon & d\lambda \end{bmatrix} (v) = \frac{1}{\varepsilon v + d\lambda}.$$

Hence $\mathcal{T}_\alpha(t,v) = (A(t), TAT(v))$ as given in Definition III.7. □

Lemma III.37. *Put $\mathcal{M} = (S^{-1}T)^{p-2}\, S^{-2}T$. For $(t,v) \in \mathcal{A} \cup \mathcal{D}_3$ we have*

$$\mathcal{T}_\alpha{}^{p-1}(t,v) = (\mathcal{M}(t), T\mathcal{M}T(v)).$$

Proof. First assume $(t,v) \in \mathcal{A}$, so $-\delta_1 \le t \le \frac{-1}{\lambda+1} < -\delta_2$. We have $\varepsilon(t) = -1$ and $d(t) = 2$, and

$$T_\alpha(t) = \frac{-1}{t} - 2\lambda.$$

We note that

$$S^{-2}T(t) = \begin{bmatrix} -2\lambda & -1 \\ 1 & 0 \end{bmatrix} (t) = \frac{-1}{t} - 2\lambda.$$

We find

$$T_\alpha(-\delta_1) = (\alpha - 1)\lambda = l_0 \quad \text{and} \quad T_\alpha\left(\frac{-1}{\lambda+1}\right) = 1 - \lambda = -H_{2p-3}.$$

As noted in the proof of Lemma III.31, one has $-H_{2p-3} < r_1$. From Theorem III.18 and the above estimates it follows that for both $-\delta_1$ and r_{p-1} the following $p-2$ applications of T_α give $\varepsilon = -1$ and $d = 1$. Thus we use $p-2$ times

$$T_\alpha(t) = \frac{-1}{t} - \lambda = \frac{-\lambda t - 1}{t} = \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix} (t) = V(t).$$

Combining the first step with these $p-2$ steps we find $\mathcal{M} = (S^{-1}T)^{p-2}S^{-2}T$ for points $(t,v) \in \mathcal{A}$. From Lemma III.36 and the fact that $TT = I$ we find that the second coordinate is given by $T\mathcal{M}T$.

Now assume $(t,v) \in \mathcal{D}_3$. In this case $\alpha \in \left(\frac{-\lambda^2+4\lambda+4}{8\lambda}, \frac{1}{\lambda}\right)$ and $-\delta_1 < r_{p-1} \leq t \leq \frac{-2}{\lambda+4} < -\delta_2$. We again have $\varepsilon(t) = -1$ and $d(t) = 2$, and find

$$T_\alpha(r_{p-1}) = \frac{2+\lambda^2(1-3\alpha)}{(2\alpha-1)\lambda} = r_p \quad \text{and} \quad T_\alpha\left(\frac{-2}{\lambda+4}\right) = 2 - \frac{3\lambda}{2}.$$

For $\alpha \in \left(\frac{-\lambda^2+4\lambda+4}{8\lambda}, \frac{1}{\lambda}\right)$ we have $r_p < 2 - \frac{3\lambda}{2} < l_1$, so like before we apply T_α in the next $p-2$ steps with $\varepsilon = -1$ and $d = 1$. □

We use the auxiliary sequence B_n from (III.8) to find powers of $S^{-1}T$.

Lemma III.38. *For $n \geq 1$ we have*

$$(S^{-1}T)^n = \begin{bmatrix} -B_{n+1} & -B_n \\ B_n & B_{n-1} \end{bmatrix}.$$

Proof. We use induction. For $n = 1$ we have

$$S^{-1}T = \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -B_2 & -B_1 \\ B_1 & B_0 \end{bmatrix}.$$

Assume that

$$(S^{-1}T)^{n-1} = \begin{bmatrix} -B_n & -B_{n-1} \\ B_{n-1} & B_{n-2} \end{bmatrix}.$$

We find

$$(S^{-1}T)^n = \begin{bmatrix} -B_n & -B_{n-1} \\ B_{n-1} & B_{n-2} \end{bmatrix} \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -B_{n+1} & -B_n \\ B_n & B_{n-1} \end{bmatrix}.$$

□

Lemma III.39. *The function ${\mathcal{T}_\alpha}^{p-1}$ is explicitly given by*

$${\mathcal{T}_\alpha}^{p-1}(t,v) = \frac{B_p}{2}\left(\begin{bmatrix} -\lambda^2-2 & -\lambda \\ \lambda^3-\lambda & \lambda^2-2 \end{bmatrix}(t), \begin{bmatrix} -\lambda^2+2 & \lambda^3-\lambda \\ -\lambda & \lambda^2+2 \end{bmatrix}(v)\right).$$

Proof. We compute $\mathcal{M}$ given in Lemma III.37 by $\mathcal{M} = (S^{-1}T)^{p-2}S^{-2}T$.

From Lemma III.38 we find

$$(S^{-1}T)^{p-2} = \begin{bmatrix} -B_{p-1} & -B_{p-2} \\ B_{p-2} & B_{p-3} \end{bmatrix}.$$

Using (III.9) and (III.8) we find

$$B_{p-1} = \frac{\lambda}{2}B_p, \quad B_{p-2} = \left(\frac{\lambda^2}{2} - 1\right)B_p \quad \text{and} \quad B_{p-3} = \left(\frac{\lambda^3}{2} - \frac{3\lambda}{2}\right)B_p.$$

So

$$(S^{-1}T)^{p-2} = \frac{B_p}{2}\begin{bmatrix} \lambda & \lambda^2-2 \\ -\lambda^2+2 & -\lambda^3+3\lambda \end{bmatrix},$$

and we find

$$\mathcal{M} = (S^{-1}T)^{p-2}S^{-2}T = \frac{B_p}{2}\begin{bmatrix} -\lambda^2-2 & -\lambda \\ \lambda^3-\lambda & \lambda^2-2 \end{bmatrix}.$$

The second coordinate is easy to calculate. □

With the explicit fomula for $\mathcal{M}(t)$ we can easily compute its fixed points, they are given by

$$t_1 = \frac{-1}{\lambda+1} \quad \text{and} \quad t_2 = \frac{-1}{\lambda-1}.$$

The fixed points of $T\mathcal{M}T(v)$ are given by

$$v_1 = \lambda+1 \quad \text{and} \quad v_2 = \lambda-1.$$

Corollary III.40. *The point* $\left(\frac{-1}{\lambda+1}, \lambda-1\right)$ *is a fixed point of* $\mathcal{T}_\alpha^{p-1}(t,v)$.

Remark III.41. If $\varepsilon=-1$ and d is constant, then $T_\alpha(t)$ is strictly increasing in t. From this and the above corollary we find that for $(t,v)\in\mathcal{A}$ we have $\mathcal{M}(t)\le t$, whilst for points $(t,v)\in\mathcal{D}_3$ we have $\mathcal{M}(t)>t$. $\diamond$

Flushing. We say a point is *flushed* when it is mapped from $\mathcal{D}$ to a point outside of $\mathcal{D}$ by $\mathcal{T}_\alpha$. We look at the flushing of points in $\mathcal{A}$ and $\mathcal{D}_3$.

Flushing from $\mathcal{A}$. Combining all the above we find that the vertices of $\mathcal{A}$ are mapped as follows under $\mathcal{T}^{p-1}$.

$$\begin{aligned}
(-\delta_1, \lambda-1) &\mapsto \left(\frac{\alpha\lambda^2-2}{(-\alpha\lambda^2+2\alpha+1)\lambda}, \lambda-1\right) = (l_{p-2}, \lambda-1)\\
\left(\frac{-1}{\lambda+1}, \lambda-1\right) &\mapsto \left(\frac{-1}{\lambda+1}, \lambda-1\right),\\
(-\delta_1, g(-\delta_1)) &\mapsto \left(l_{p-2}, \lambda-\frac{(2\alpha-1)\lambda-4}{(\alpha\lambda-2)\lambda-2}\right).
\end{aligned}$$

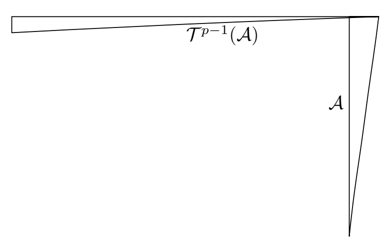


FIGURE 6. $\mathcal{A}$ and its $(p-1)$st transformation under $\mathcal{T}_\alpha$.

The image of $\mathcal{A}$ under $\mathcal{T}_\alpha^{p-1}$ is a long, thin "triangle" that has a "triangular" intersection with $\mathcal{A}$; see Figure 6. We notice in particular that $(-\delta_1, H_{2p-3})$ is included in $\mathcal{T}_\alpha^{p-1}(\mathcal{A})$. However, the part of $\mathcal{T}_\alpha^{p-1}(\mathcal{A})$ on the left-hand side of the line $t=-\delta_1$ is in Region (II). So these points are flushed in the next application of $\mathcal{T}_\alpha$.

We conclude that for the points $(t_n, v_n)\in\mathcal{A}$ with $T_\alpha^{p-1}(t_n)<-\delta_1$ we have

$$\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}, \ldots, \Theta_{n+p-1} > \mathcal{H}_q\} \quad \text{and} \quad \Theta_{n+p} < \mathcal{H}_q.$$

The following theorem generalizes this idea to multiple rounds. Recall from (III.19) that we define τ_k by

$$(\tau_k, \nu_k) = \mathcal{T}_\alpha^{-k(p-1)}(-\delta_1, H_1).$$

Theorem III.42. *Let $k \geq 1$ be an integer.*

Any point (t, v) of $\mathcal{A}$ is flushed after exactly k rounds if and only if

$$\tau_{k-1} \leq t < \tau_k.$$

For any x with $\tau_{k-1} \leq t_n < \tau_k$,

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)-1}, \Theta_{n+k(p-1)}\} > \mathcal{H}_q,$$

while

$$\Theta_{n+k(p-1)+1} < \mathcal{H}_q.$$

Proof. A point $(t, v) \in \mathcal{A}$ is flushed after exactly k rounds if k is minimal such that $\mathcal{T}_\alpha^{k(p-1)}(t, v)$ has its first coordinate smaller than $-\delta_1$. The result follows from the definition of τ_k and the above. $\square$

Flushing from $\mathcal{D}_3$. Recall that the vertices of $\mathcal{D}_3$ are given by $(r_{p-1}, g(r_{p-1}))$, $\left(\frac{-2}{\lambda+4}, \frac{\lambda}{2}\right)$ and $(r_{p-1}, \frac{\lambda}{2})$; see Figure 5. We have $\frac{-1}{\lambda+1} < r_{p-1} < \frac{-2}{\lambda+4}$ and $\lambda - 1 < g(r_{p-1}) < \frac{\lambda}{2}$. The fixed point $\left(\frac{-1}{\lambda+1}, \lambda - 1\right)$ of $\mathcal{T}^{p-1}$ is repelling in the t-direction and attractive in the v-direction.

Lemma III.43. *There exists a positive integer K such that all points in $\mathcal{D}_3$ are flushed after K rounds.*

Proof. Let $(t_1, \lambda - 1 + v_1)$ be a point in $\mathcal{D}_3$, it follows that $0 < v_1 < 1$. With Lemma III.39 we find for the second coordinate

$$\pi_2(\mathcal{T}_\alpha^{p-1}(t_1, \lambda-1+v_1)) = \frac{(-\lambda^2+2)(\lambda-1+v_1)+\lambda^3-\lambda}{-\lambda(\lambda-1+v_1)+\lambda^2+2} = \lambda-1+\frac{2-\lambda}{\lambda+2-\lambda v_1}v_1.$$

As $\frac{2-\lambda}{\lambda+2-\lambda v_1} < 1 - \frac{\lambda}{2} < 1$, the result follows. $\square$

Remark III.44. We could divide $\mathcal{D}_3$ in parts that get flushed after $1, 2, \ldots, K$ rounds, respectively. As we saw in the example for $q = 4$ the formulas needed to do this are rather ugly and in this general case they only get worse. For our main result we only need that after finitely many rounds all points are flushed out of $\mathcal{D}_3$. $\diamond$

We still need to consider points in Region (I). It follows from Theorem III.18 that after at most $p - 2$ steps each such point is either flushed or mapped into $\mathcal{A} \cup \mathcal{D}_3$. We are now ready to prove Theorem III.20 for this case.

Proof of Theorem III.20 for $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$. By definition $\tau_k < \tau_{k+1}$. In [**8**] it was shown that $\frac{-1}{\lambda+1} = [\overline{(-1:2), (-1:1)^{p-2}}]$ and we find $\lim_{k\to\infty} \tau_k = \dfrac{-1}{\lambda+1}$. Recall that $c_k = \dfrac{-\tau_{k-1}}{1+\tau_{k-1}\nu_{k-1}}$. It follows that $c_{k+1} < c_k$ and $\lim_{k\to\infty} c_k = \dfrac{1}{2}$.

Take an integer k such that $k > K$ from Lemma III.43. Take a point $(t_n, v_n) \in \mathcal{D}$ that did not get flushed in the first $k - 1$ rounds. There exists an index i with $0 \leq i \leq p - 2$ such that (t_{n+i}, v_{n+i}) is either in $\mathcal{A}$ or is flushed. We assume

$(t_{n+i}, v_{n+i}) \in \mathcal{A}$, otherwise we are done. From Theorem III.42 we find that $t_{n+i} \geq \tau_k$. Thus $\Theta_n(t_{n+i}, v_{n+i}) \leq \Theta_n(\tau_{k-1}, \lambda - 1) = c_k$. $\square$

3.2. Even case for $\alpha = \frac{1}{\lambda}$. The natural extension $\Omega_{1/\lambda}$ can be found from $\Omega_{1/2}$ by reflecting in the line $v = -t$ if $t \leq 0$, and in the line $v = t$ if $x \geq 0$; see the remark on page 9 of [DKS].

In the case $\alpha = \frac{1}{2}$ we have $l_n = r_n$ for $n \geq 1$, see Theorem III.18. We put $\varphi_0 = l_0 = -\frac{\lambda}{2}$ and denote $\varphi_n = l_n = r_n = T_{1/2}^n(\varphi_0)$. Put

$$L_1 = \frac{1}{\lambda + 1} \quad \text{and} \quad L_n = \frac{1}{\lambda - L_{n-1}} \quad \text{for} \quad n = 2, 3, \ldots, p-1.$$

We know from [**11**] that

$$\Omega_{1/2} = \left(\bigcup_{n=1}^{p-1} [\varphi_{n-1}, \varphi_n) \times [0, L_n] \right) \cup [0, -\varphi_0) \times [0, 1],$$
$$\Omega_{1/\lambda} = \left(\bigcup_{n=1}^{p-2} [-L_{p-n}, -L_{p-n-1}) \times [0, -\varphi_{p-n-1}] \right) \cup [-L_1, 1) \times [0, -\varphi_0],$$

see Figure 7 for an example with $q = 8$.

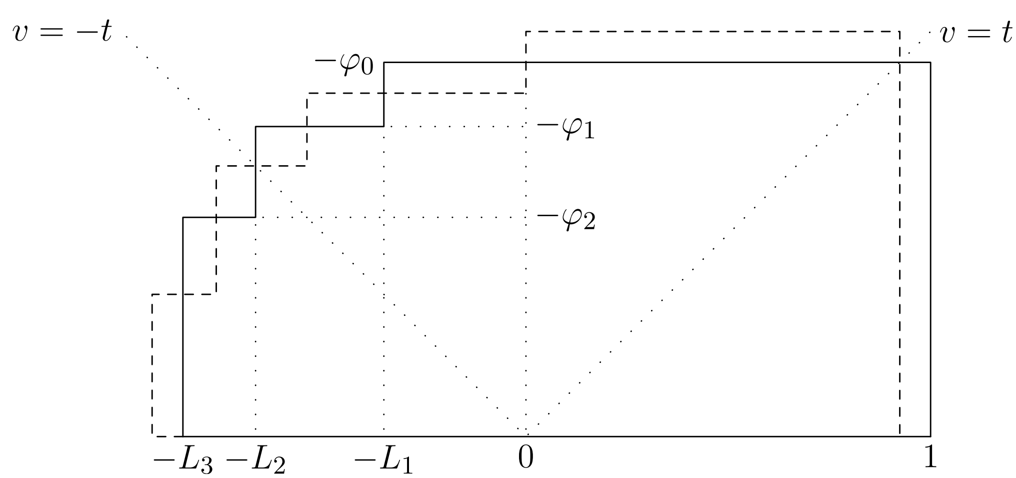


FIGURE 7. We find $\Omega_{\frac{1}{\lambda}}$ by reflecting $\Omega_{\frac{1}{2}}$ (dashed) in $v = |t|$. In this example $q = 8$.

We use the following lemma to derive the shape of

$$\mathcal{D} = \left\{ (t, v) \in \Omega_{1/\lambda} | \min \left\{ \frac{v}{1 + tv}, \frac{|t|}{1 + tv} \right\} > \frac{1}{2} \right\}.$$

Lemma III.45. *Under reflection in the lines $v = |t|$, the graph of f is the image of the graph of g and vice versa.*

Proof. If $t \le 0$, the points on the graph of $f(t)$ are reflected to points $(-f(t), -t) = \left(\frac{-1}{2-t}, -t\right)$. Write $t' = \frac{-1}{2-t}$. The reflection can be written as $\left(t', \frac{-2t'-1}{t'}\right) = (t', g(t'))$.

If $t \ge 0$, the points on the graph of $f(t)$ are reflected to points $(f(t), t) = \left(\frac{1}{2-t}, t\right)$. Write $t' = \frac{1}{2-t}$. The reflection can be written as $\left(t', \frac{2t'-1}{t'}\right) = (t', g(t'))$ (and vice versa in both cases). $\square$

Thanks to the reflecting it is easy to describe the shape of $\mathcal{D} = \mathcal{D}_{1/\lambda}$ in $\Omega_{1/\lambda}$; it is simply the reflection of $\mathcal{D}_{1/2}$ in $\Omega_{\frac{1}{2}}$, also see Figure 8.

Lemma III.46. [30] *For all even q, $\mathcal{D}_{1/2}$ consists of two components $\mathcal{D}_{1/2,1}$ and $\mathcal{D}_{1/2,2}$. The subregion $\mathcal{D}_{1/2,1}$ is bounded by the lines $t = \varphi_0, v = L_1$ and the graph of f; $\mathcal{D}_{1/2,2}$ is bounded by the graph of g from the right, by the graph of f from below and by the boundary of $\Omega_{1/2}$.*

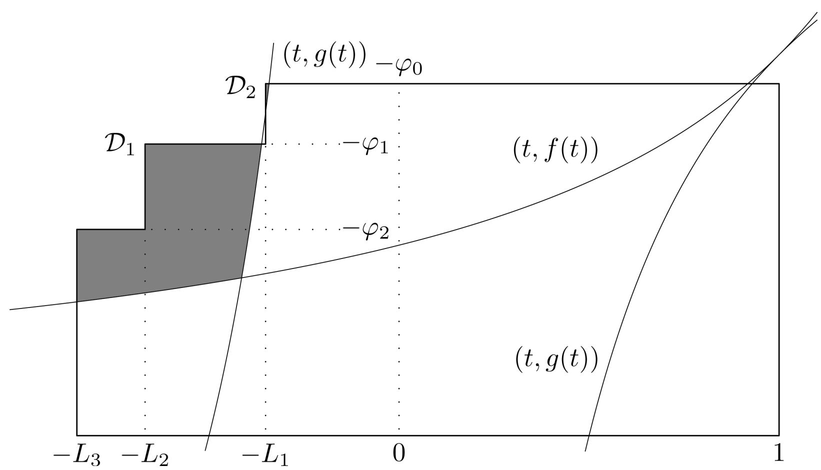


Figure 8. The graphs of f and g in $\Omega_{1/\lambda}$ for $q = 8$. We shaded the area $\mathcal{D}$ grey. The bigger part on the left is called $\mathcal{D}_1$, the small 'triangle' is $\mathcal{D}_2$.

Corollary III.47. *For all even q the region $\mathcal{D}_{1/\lambda}$ consists of two components $\mathcal{D}_1$ and $\mathcal{D}_2$. The area $\mathcal{D}_1$ is bounded by $t = -L_{p-1}$ on the left, the boundary of $\Omega_{1/\lambda}$ and the graphs of f and g. The other area $\mathcal{D}_2$ is bounded by the lines $t = -L_1$ and $v = -\varphi_0$ and the graph of g.*

Remark III.48. We have $\mathcal{D}_1 = \mathcal{D}^T_{1/2,2}$ and $\mathcal{D}_2 = \mathcal{D}^T_{1/2,1}$. $\diamond$

We now could proceed as in the previous two subsections; For $t \ge 0$ one easily sees that the graphs of f and g do not meet in $\Omega_{1/\lambda}$ (they meet in $(1, 1)$, which is outside $\Omega_{1/\lambda}$). As before, from this it follows that for $t_n \ge 0$ we have that $\min\{\Theta_{n-1}, \Theta_n\} < \frac{1}{2}$. So we only need to focus on the region $\mathcal{D}$, and how it is eventually "flushed". However, we can also derive the result directly from the case $\alpha = \frac{1}{2}$.

Define the map $M : \Omega_{1/\lambda} \to \Omega_{1/2}$ by

$$M(t,v) = \begin{cases} (-v,-t) & \text{if } t < 0, \\ (v,t) & \text{if } t \geq 0. \end{cases} \tag{III.49}$$

In [**11**] it was shown that

$$\mathcal{T}_{1/\lambda}(t,v) = M^{-1}\left(\mathcal{T}_{1/2}^{-1}(M(t,v))\right). \tag{III.50}$$

This implies that the dynamical systems $(\Omega_{1/2}, \mu_{1/2}, \mathcal{T}_{1/2})$ and $(\Omega_{1/\lambda}, \mu_{1/\lambda}, \mathcal{T}_{1/\lambda})$ are isomorphic. These systems "behave dynamically in the same way", also see Chapter IV. Usually, this is not much of help if we want to obtain Diophantine properties of one system from the other system. But the special form of the isomorphism M makes it possible to prove directly that these systems possess the same "Diophantine properties." Essentially, if one "moves forward in time" in $(\Omega_{1/2}, \mu_{1/2}, \mathcal{T}_{1/2})$, then one "moves backward in time" in $(\Omega_{1/\lambda}, \mu_{1/\lambda}, \mathcal{T}_{1/\lambda})$ and vice versa.

Theorem III.51. *Let $\ell \in \mathbb{N}$, and let $\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+\ell}$ be $\ell + 2$ consecutive approximation coefficients of the point $(t_n, v_n) \in \Omega_{1/\lambda}$, then there exists a point $(\tilde{t}_m, \tilde{v}_m) \in \Omega_{1/2}$ and approximation coefficients $\tilde{\Theta}_m, \tilde{\Theta}_{m-1}, \ldots, \tilde{\Theta}_{m-\ell-1}$, given by*

$$\tilde{\Theta}_m = \Theta_m(\tilde{t}_m, \tilde{v}_m) = \frac{|\tilde{t}_m|}{1 + \tilde{t}_m \tilde{v}_m}, \quad \tilde{\Theta}_{m-1} = \Theta_{m-1}(\tilde{t}_m, \tilde{v}_m) = \frac{\tilde{v}_m}{1 + \tilde{t}_m \tilde{v}_m}, \ldots,$$

$$\tilde{\Theta}_{m-\ell-1} = \Theta_{m-\ell-1}(\tilde{t}_{m-\ell}, \tilde{v}_{m-\ell}) = \frac{\tilde{v}_{m-\ell}}{1 + \tilde{t}_{m-\ell} \tilde{v}_{m-\ell}},$$

such that

$$\Theta_{n-1} = \tilde{\Theta}_m,\ \Theta_n = \tilde{\Theta}_{m-1},\ \Theta_{n+1} = \tilde{\Theta}_{m-2},\ \ldots,\ \Theta_{n+\ell} = \tilde{\Theta}_{m-\ell-1}. \tag{III.52}$$

Proof. Let $(\tilde{t}_m, \tilde{v}_m) = M(t_n, v_n) \in \Omega_{1/2}$ be the point in $\Omega_{1/2}$ corresponding to $(t_n, v_n) \in \Omega_{1/\lambda}$ under the isomorphism M from (III.49). Then

$$(\tilde{t}_m, \tilde{v}_m) = \begin{cases} (-v_n, -t_n) & \text{if } t_n < 0, \\ (v_n, t_n) & \text{if } t_n \geq 0, \end{cases}$$

and we find that

$$\tilde{\Theta}_{m-1} = \Theta_{m-1}(\tilde{t}_m, \tilde{v}_m) = \frac{\tilde{v}_m}{1 + \tilde{t}_m \tilde{v}_m} = \frac{|t_n|}{1 + t_n v_n} = \Theta_n.$$

Similarly,

$$\tilde{\Theta}_m = \Theta_m(\tilde{t}_m, \tilde{v}_m) = \frac{|\tilde{t}_m|}{1 + \tilde{t}_m \tilde{v}_m} = \frac{v_n}{1 + t_n v_n} = \Theta_{n-1}.$$

So we have $(\Theta_{n-1}, \Theta_n) = (\tilde{\Theta}_m, \tilde{\Theta}_{m-1})$.

Furthermore, by (III.50) we have that

$$\begin{aligned} (t_{n+1}, v_{n+1}) &= \mathcal{T}_{1/\lambda}(t_n, v_n) = M^{-1}\left(\mathcal{T}_{1/2}(M(t_n, v_n))\right) \\ &= M^{-1}\left(\mathcal{T}_{1/2}(\tilde{t}_m, \tilde{v}_m)\right) = M^{-1}(\tilde{t}_{m-1}, \tilde{v}_{m-1}), \end{aligned}$$

and we see that

$$(\tilde{t}_{m-1}, \tilde{v}_{m-1}) = M\,(t_{n+1}, v_{n+1}) = \begin{cases} (-v_{n+1}, -t_{n+1}) & \text{if } t_{n+1} < 0 \\ (v_{n+1}, t_{n+1}) & \text{if } t_{n+1} \geq 0. \end{cases}$$

But then we have that

$$\tilde{\Theta}_{m-2} = \frac{\tilde{v}_{m-1}}{1 + \tilde{t}_{m-1}\tilde{v}_{m-1}} = \frac{|t_{n+1}|}{1 + t_{n+1}v_{n+1}} = \Theta_{n+1}.$$

By induction it follows that

$$\Theta_{n-1} = \tilde{\Theta}_m,\ \Theta_n = \tilde{\Theta}_{m-1},\ \Theta_{n+1} = \tilde{\Theta}_{m-2},\ \ldots,\ \Theta_{n+\ell} = \tilde{\Theta}_{m-\ell-1}.$$

□

Lemma III.53. *Let Θ_{n-1} and Θ_n be two consecutive approximation coefficients of the point (t_n, v_n). Then*

$$t_n = \frac{1 + \varepsilon_{n+1}\sqrt{1 - 4\varepsilon_{n+1}\Theta_{n-1}\Theta_n}}{2\Theta_{n-1}} \quad \textit{and} \quad v_n = \frac{\varepsilon_{n+1} + \sqrt{1 - 4\varepsilon_{n+1}\Theta_{n-1}\Theta_n}}{2\Theta_n}.$$

Proof. From (III.12) we have $\Theta_{n-1} = \dfrac{v_n}{1 + t_n v_n}$ and $\Theta_n = \dfrac{\varepsilon_{n+1} t_n}{1 + t_n v_n}$. It follows that

$$v_n = \frac{\varepsilon_{n+1}\Theta_{n-1}\, t_n}{\Theta_n}, \tag{III.54}$$

and substituting (III.54) in the formula for Θ_n yields

$$\varepsilon_{n+1}\Theta_{n-1}t_n^2 - \varepsilon_{n+1}t_n + \Theta_n = 0,$$

from which we find t_n. Substituting t_n in (III.54) yields v_n. □

Proof of Theorem III.20 for $\alpha = 1/\lambda$. Let x be a G_q-irrational with $1/\lambda$-expansion $[\,\varepsilon_1 : d_1, \varepsilon_2 : d_2, \varepsilon_3 : d_3, \ldots]$ and let $n \geq 1$ be an integer. Assume there exists a $k \in \mathbb{N}$ such, that

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)}\} > \frac{1}{2},$$

otherwise we are done. From Lemma III.53 we find the appropriate $(t_n, v_n) \in \Omega_{1/\lambda}$ for this sequence of approximation coefficients. From Theorem III.51 if follows that we can find $(\tilde{t}_m, \tilde{v}_m) \in \Omega_{1/2}$ such that

$$\Theta_{n-1} = \tilde{\Theta}_m, \Theta_n = \tilde{\Theta}_{m-1}, \ldots, \Theta_{n+k(p-1)} = \tilde{\Theta}_{m-k(p-1)-1}.$$

It follows from Theorem III.2 for $\alpha = \frac{1}{2}$ that

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)}\} = \min\{\tilde{\Theta}_{m-k(p-1)-1}, \ldots, \tilde{\Theta}_{m-1}, \tilde{\Theta}_m,\} < c_k,$$

where c_k is defined in (III.19) and Theorem III.20. This proves Theorem III.20 for the case $\alpha = 1/\lambda$. □

4. Tong's spectrum for odd α-Rosen fractions

Let $q = 2h+3$ for $h \geq 1$ and define

$$\rho = \frac{\lambda - 2 + \sqrt{\lambda^2 - 4\lambda + 8}}{2}. \tag{III.55}$$

Remark III.56. We often use the following relations for ρ

$$\rho^2 + (2-\lambda)\rho - 1 = 0 \quad \text{and} \quad \frac{\rho}{\rho^2+1} = \mathcal{H}_q = \frac{1}{\sqrt{\lambda^2 - 4\lambda + 8}}.$$

◇

Remark III.57. For $q = 3$ (i.e. $\lambda = 1$) we are in the "classical" case of Nakada's α-expansions [**41**]. In this case $\frac{\rho}{\lambda} = \frac{\sqrt{5}-1}{2} = g$ and $\mathcal{H}_q = \frac{1}{\sqrt{5}}$, see also [**17**] for a discussion of this case. ◇

For a fixed λ we define the following constants

$$\begin{aligned}
\alpha_1 &= \frac{(\lambda-2)\mathcal{H}_q + 1}{\lambda}, \\
\alpha_2 &= \frac{-\lambda + \sqrt{5\lambda^2 - 4\lambda + 4}}{2\lambda}, \\
\alpha_3 &= \frac{(2-\lambda)^2\mathcal{H}_q + 2\lambda}{4\lambda}, \\
\alpha_4 &= \frac{(2-\lambda)\mathcal{H}_q - 2}{(\lambda\mathcal{H}_q - 2\mathcal{H}_q - 2)\lambda}.
\end{aligned}$$

For all admissible λ we have

$$\frac{1}{2} < \alpha_1 < \alpha_2 < \alpha_3 < \frac{\rho}{\lambda} < \alpha_4 < \frac{1}{\lambda}. \tag{III.58}$$

Remark III.59. These numbers are very near to each other. For example, if $q = 9$,

$$\begin{aligned}
&\alpha_1 \approx 0.500058, \qquad \alpha_2 \approx 0.500515, \quad \alpha_3 \approx 0.500966 \\
&\frac{\rho}{\lambda} \approx 0.500967, \qquad \alpha_4 \approx 0.500994, \quad \frac{1}{\lambda} \approx 0.532089.
\end{aligned}$$

◇

In [**11**] it was shown that there are four subcases for the natural extension of odd α-Rosen fractions: $\alpha = \frac{1}{2}, \alpha \in \left(\frac{1}{2}, \frac{\rho}{\lambda}\right), \alpha = \frac{\rho}{\lambda}$ and $\alpha \in \left(\frac{\rho}{\lambda}, \frac{1}{\lambda}\right]$. Again, the case $\alpha = \frac{1}{2}$ had been dealt with in [**30**] and we give the details for the other cases in this section.

The following theorem from [**11**] is the counterpart for the odd case of Theorem III.18.

Theorem III.60. *Let $q = 2h+3, h \in N, h \geq 1$. We have the following cases.*

$\alpha = \frac{1}{2}$: $l_0 < r_{h+1} = l_{h+1} < r_1 = l_1 < \ldots < r_{h+n} = l_{h+n} < r_n = l_n < \ldots < r_{2h-1} = l_{2h-1} < r_{h-1} = l_{h-1} < r_{2h} = l_{2h} < -\delta_1 < r_h = l_h < -\delta_2 < r_{2h+1} = l_{2h+1} = 0 < r_0$.

$\frac{1}{2} < \alpha < \frac{\rho}{\lambda}$: $l_0 < r_{h+1} < l_{h+1} < r_1 < l_1 < \ldots < r_{h+n} < l_{h+n} < r_n < l_n < \ldots < r_{2h-1} < l_{2h-1} < r_{h-1} < l_{h-1} < r_{2h} < l_{2h} < -\delta_1 < r_h < l_h < r_{2h+1} < 0 < l_{2h+1} < r_0$.
Furthermore, we have $l_h < -\delta_2$, $l_{2h+2} = r_{2h+2}$ *and* $d_{2h+2}(r_0) = d_{2h+2}(l_0)+1$.

$\alpha = \frac{\rho}{\lambda}$: $l_0 = r_{h+1} < l_{h+1} = r_1 < \ldots < l_{n-1} = r_{h+n} < l_{h+n} = r_n < \ldots < l_{h-1} = r_{2h} < l_{2h} = r_h = -\delta_1 < l_h = r_{2h+1} < -\delta_2 < 0 < r_0$.

$\frac{\rho}{\lambda} < \alpha < \frac{1}{\lambda}$: $l_0 < r_1 < l_1 < r_2 < \ldots < l_{h-1} < r_h < -\delta_1 < l_h < 0 < r_{h+1} < r_0$.
Furthermore, we have $l_{h+1} = r_{h+2}$ *and* $d_{h+1}(l_0) = d_{h+2}(r_0) + 1$.

$\alpha = \frac{1}{\lambda}$: $l_0 = r_1 < l_1 = r_2 < \ldots < l_{h-1} = r_h < -\delta_1 < l_h = 0 = r_{h+1} < r_0$.

Remark III.61. In a preliminary version of [**11**] there was a small error in the above theorem in the case $\frac{1}{2} < \alpha < \frac{\rho}{\lambda}$: it stated that $-\delta_2 < r_{2h+1}$. But this is only true if $\alpha < \alpha_2$. For all $\frac{1}{2} < \alpha < \frac{\rho}{\lambda}$ one has $r_{2h+1} = -\frac{(2\alpha-1)\lambda}{\alpha\lambda^2-2\lambda+2}$, so

$$\begin{aligned} r_{2h+1} \geq -\delta_2 &\Leftrightarrow -\frac{(2\alpha-1)\lambda}{\alpha\lambda^2 - 2\lambda + 2} \geq \frac{-1}{(\alpha+2)\lambda} \\ &\Leftrightarrow \frac{-\lambda - \sqrt{5\lambda^2 - 4\lambda + 4}}{2\lambda} < \alpha \leq \frac{-\lambda + \sqrt{5\lambda^2 - 4\lambda + 4}}{2\lambda} = \alpha_2. \end{aligned}$$

We conclude that $r_{2h+1} \geq -\delta_2$ if $\alpha \in \left(\frac{1}{2}, a_2\right)$ and that $r_{2h+1} < -\delta_2$ if $\alpha \in \left[a_2, \frac{\rho}{\lambda}\right)$. $\diamond$

In this section we prove the following result.

Theorem III.62. *Fix an odd* $q = 2h+3$, *with* $h \geq 1$.

(i) Let $\alpha \in \left[\frac{1}{2}, \frac{\rho}{\lambda}\right]$. *Then there exists a positive integer* K *such that for every* G_q*-irrational number* x *and all positive* n *and* $k > K$,

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(2h+1)}\} < c_k,$$

for certain constants c_k *with* $c_{k+1} < c_k$ *and* $\lim_{k\to\infty} c_k = \mathcal{H}_q$.

(ii) Let $\alpha \in \left(\frac{\rho}{\lambda}, \frac{1}{\lambda}\right]$. *For every* G_q*-irrational number* x *and all positive* n, *one has*

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+(3h+2)}\} < \mathcal{H}_q.$$

As Theorem III.60 already suggests, the behavior of the dynamical system drastically changes at the point $\alpha = \rho/\lambda$. Case (i) is similar to the even case, but case (ii) yields a finite spectrum. We note such a finite spectrum was already described for the case $q = 3$ in [**17**].

For odd $q = 2h+3$ we define a *round* by $2h+1$ consecutive applications of $\mathcal{T}_\alpha$.

Intersection of the graphs of $f(t)$ and $g(t)$. We start by looking at the behavior of the graphs of $f(t)$ and $g(t)$ as given in (III.15) on Ω_α for odd q.

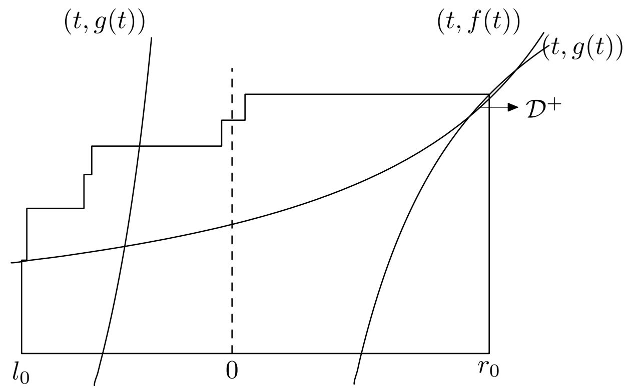


FIGURE 9. For $\alpha > \rho/\lambda$ we have a part $\mathcal{D}^+$ for positive t.

In case $t > 0$ we find that

$$f(t) = g(t) \quad \text{if and only if} \quad t = \frac{1 \pm \sqrt{1 - 4\mathcal{H}_q^2}}{2\mathcal{H}_q}.$$

Since $\mathcal{H}_q < 1/2$,

$$\frac{1 + \sqrt{1 - 4\mathcal{H}_q^2}}{2\mathcal{H}_q} > 1 \geq \alpha\lambda$$

for all $\alpha \in [1/2, 1/\lambda]$, so we only need to consider $\frac{1-\sqrt{1-4\mathcal{H}_q^2}}{2\mathcal{H}_q} = \rho$.

Note that $f(\rho) = \rho$. In case

$$\alpha = \frac{1 - \sqrt{1 - 4\mathcal{H}_q^2}}{2\lambda\mathcal{H}_q} = \frac{\rho}{\lambda}, \tag{III.63}$$

we find that for $t > 0$ the intersection of the graphs of f and g is on the boundary of Ω_α, so if $\alpha \in [1/2, \rho/\lambda)$ we have that the intersection point for $t > 0$ is outside Ω_α, while for $\alpha \in (\rho/\lambda, 1/\lambda]$ it is inside Ω_α. Therefore, for the latter values of α we have an extra part $\mathcal{D}^+$ for positive t; see Figure 9. In this case we denote $\mathcal{D}^- = \{(x, y) \in \mathcal{D} \mid x \leq 0\}$ and $\mathcal{D}^+ = \{(x, y) \in \mathcal{D} \mid x > 0\}$.

4.1. Odd case for $\alpha \in (\frac{1}{2}, \frac{\rho}{\lambda})$. In this case the natural extension is given by $\Omega_\alpha = \bigcup_{n=1}^{4h+3} J_n \times [0, H_n]$, with

$$\begin{aligned} &J_{4n-3} = [l_{n-1}, r_{h+n}), \quad J_{4n-2} = [r_{h+n}, l_{h+n}) \qquad \text{for} \quad n = 1, \ldots, h+1, \\ &J_{4n-1} = [l_{h+n}, r_n), \qquad J_{4n} = [r_n, l_n) \qquad\qquad \text{for} \quad n = 1, \ldots h, \end{aligned}$$

and

$$H_1 = \frac{1}{\lambda + 1/\rho}, \quad H_2 = \frac{1}{\lambda + 1}, \quad H_3 = \frac{1}{\lambda + \rho}, \quad H_4 = \frac{1}{\lambda},$$

$$\text{and} \quad H_n = \frac{1}{\lambda - H_{n-4}} \quad \text{for} \quad n = 5, 6, \ldots, 4h+3.$$

In [**11**] is shown that $H_{4h-1} = \lambda - \frac{1}{\rho}, H_{4h} = \lambda - 1, H_{4h+1} = \lambda - \rho$ and $H_{4h+2} = \frac{\lambda}{2}$. Furthermore, from [**11**] we have that

$$l_h = \frac{1 - \alpha\lambda}{(\lambda - 1)\alpha\lambda - 1}, \quad r_h = -\frac{1 - (1-\alpha)\lambda}{1 - (1-\alpha)\lambda(\lambda - 1)} \quad \text{and}$$
$$r_{2h+1} = -\frac{(2\alpha - 1)\lambda}{\alpha\lambda^2 - 2\lambda + 2}.$$

We define $\mathcal{D}$ as in (III.17). We have proved above that in case $\alpha \leq \rho/\lambda$ we have $\mathcal{D}^+ = \emptyset$. We could divide $\mathcal{D}^-$ into regions where d_n, ε_{n+1} and ε_{n+2} are constant, as we did in Section 3.1.2. However, like before the region where $d_n = 2$ is the crucial one and we only describe the subregion of $\mathcal{D}^-$ that is on the right hand side of the line $t = -\delta_1$.

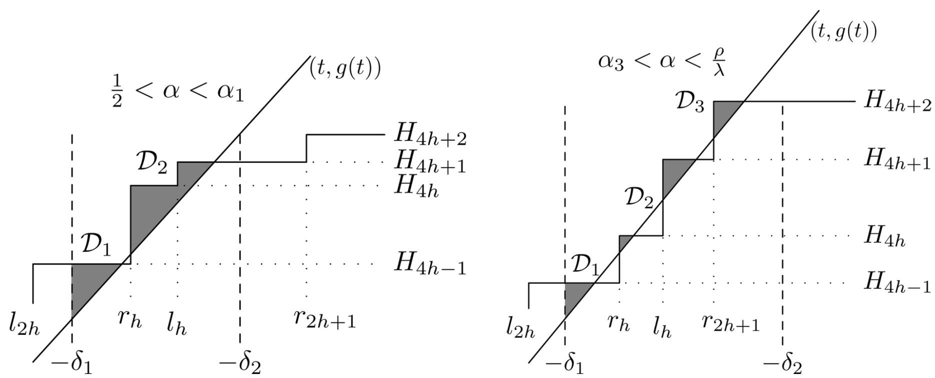


FIGURE 10. Schematic presentation of the part of $\mathcal{D}^-$ with $t > -\delta_1$ in two cases. If $\alpha_1 \leq \alpha < \alpha_2$, we have a similar picture as on the left: the only difference is that $\mathcal{D}_2$ is split into two component in this case. If $\alpha_2 < \alpha \leq \alpha_3$ the picture is similar to the one on the right, but there is no $\mathcal{D}_3$ in this case.

Lemma III.64. *If $\alpha \in \left(\frac{1}{2}, \alpha_1\right)$, then the subregion of $\mathcal{D}$ with $-\delta_1 < t < 0$ consists of two components: $\mathcal{D}_1$ bounded by the line $t = -\delta_1$ from the left, the line $v = H_{4h-1}$ from above and the graph of g from below. The second component $\mathcal{D}_2$ is bounded by line segments with $t = r_h, v = H_{4h}, t = l_h, v = H_{4h+1}$ and by the graph of g.*

If $\alpha \in [\alpha_1, \alpha_3]$, then $\mathcal{D}_1$ is as in the above case, but $\mathcal{D}_2$ is split into two components bounded by line segments as above and by the graph of g.

If $\alpha \in \left(\alpha_3, \frac{\rho}{\lambda}\right)$, then $\mathcal{D}_1$ and $\mathcal{D}_2$ are as in the previous case, but there is also a component $\mathcal{D}_3$, bounded by the line $t = r_{2h+1}$ from the left, the line $v = H_{4h+2}$ from above and the graph of g from below.

PROOF. Recall that $H_{4h-1} = \lambda - \frac{1}{\rho}, H_{4h} = \lambda - 1, H_{4h+1} = \lambda - \rho$ and $H_{4h+2} = \frac{\lambda}{2}$. First of all $g(l_h) = -\frac{1}{\mathcal{H}_q} - \frac{1}{l_h} = -\frac{1}{\mathcal{H}_q} - \frac{(\lambda-1)\alpha\lambda - 1}{1 - \alpha\lambda}$ and we find

$$H_{4h-1} < g(l_h) < \quad H_{4h} \qquad \text{if} \quad \frac{1}{2} < \alpha < \alpha_1,$$
$$H_{4h} \leq g(l_h) < \quad H_{4h+1} \qquad \text{if} \quad \alpha_1 \leq \alpha < \frac{\rho}{\lambda}.$$

From $g(r_{2h+1}) = -\frac{1}{\mathcal{H}_q} + \frac{\alpha\lambda^2-2\lambda+2}{(2\alpha-1)\lambda}$ we find that

$$g(r_{2h+1}) > H_{4h+2} \quad \text{if and only if} \quad \alpha < \frac{(2-\lambda)^2\mathcal{H}_q + 2\lambda}{4\lambda} = \alpha_3.$$

Finally, for all $\alpha \in \left(\frac{1}{2}, \frac{\rho}{\lambda}\right)$ we have that

$$g(-\delta_1) < H_{4h-1} < g(r_h) < H_{4h}, \quad g(r_{2h+1}) > H_{4h+1} \quad \text{and} \quad g(-\delta_2) > H_{4h+2},$$

which finishes the proof. $\square$

The proof of Theorem III.62 for $\frac{1}{2} \leq \alpha < \frac{\rho}{\lambda}$. Consider the intersection point of the graph of g with the line $v = H_{4h+1} = \lambda - \rho$. The first coordinate of this point is given by

$$t_1 = \frac{-\mathcal{H}_q}{1+(\lambda-\rho)\mathcal{H}_q} = \frac{-\rho}{1+\rho\lambda}.$$

Note that $t_1 \in (l_h, -\delta_2)$. We find $T_\alpha(t_1) = \frac{1}{\rho} - \lambda$ and it easily follows that we have $T_\alpha(t_1) \in (l_{h+1}, r_1)$ for all $\alpha < \frac{\rho}{\lambda}$. From Theorem III.60 we conclude that $T_\alpha^h(t_1) = (S^{-1}T)^{h-1}S^{-2}T(t_1)$. From Lemma III.38 and relations (III.8) and (III.10) we get

$$\begin{aligned}(S^{-1}T)^{h-1}S^{-2}T &= \begin{bmatrix} -B_h & -B_{h-1} \\ B_{h-1} & B_{h-2} \end{bmatrix} \begin{bmatrix} -2\lambda & -1 \\ 1 & 0 \end{bmatrix} \\ &= B_{h+1} \begin{bmatrix} -\lambda+1 & -\lambda^2+\lambda+1 \\ \lambda^2-\lambda-1 & \lambda^3-\lambda^2-2\lambda+1 \end{bmatrix} \begin{bmatrix} -2\lambda & -1 \\ 1 & 0 \end{bmatrix} \\ &= B_{h+1} \begin{bmatrix} \lambda^2-\lambda+1 & \lambda-1 \\ -\lambda^3+\lambda^2+1 & -\lambda^2+\lambda+1 \end{bmatrix}.\end{aligned}$$

We find $T_\alpha^h(t_1) = -\dfrac{\lambda-\rho-1}{\lambda(\lambda-\rho-1)+\rho-1}$. And using $\rho^2+(2-\lambda)\rho-1=0$ we find

$$g(T_\alpha^h(t_1)) = \frac{-1}{\mathcal{H}_q} - \frac{1}{T_\alpha^h(t_1)} = -\rho - \frac{1}{\rho} + \lambda + \frac{\rho-1}{\lambda-\rho-1} = \lambda - \frac{1}{\rho} = H_{4h-1}.$$

So $(T_\alpha^h(t_1), \lambda - \frac{1}{\rho})$ is the intersection point of the graph of g with the height $v = H_{4h-1}$. From the proof of Lemma III.64 it follows that $T_\alpha^h(t_1) \in (-\delta_1, r_h)$. We find that

$$T_\alpha^{h+1}(t_1) = T_\alpha(T_\alpha^h(t_1)) = -\lambda + \frac{\rho-1}{\lambda-\rho-1}.$$

Since $T_\alpha^{h+1}(t_1) < r_{h+1}$, we conclude that

$$\begin{aligned}T_\alpha^{2h+1}(t_1) &= (S^{-1}T)^h S^{-2}T(S^{-1}T)^{h-1}S^{-2}T(t_1) \\ &= B_{h+1} \begin{bmatrix} -1 & -\lambda+1 \\ \lambda-1 & \lambda^2-\lambda-1 \end{bmatrix} \left(-\lambda + \frac{\rho-1}{\lambda-\rho-1}\right) \\ &= \frac{\lambda-2\rho}{2+\lambda(\rho-2)} = \frac{-\rho}{1+\lambda\rho} = t_1.\end{aligned}$$

We find that $(t_1, \lambda-\rho)$ is a fixed point of $\mathcal{T}_\alpha^{2h+1}$.

The rest of the proof is similar to the even case for $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$. In this case $(\tau_k, \nu_k) = \mathcal{T}_\alpha^{-k(2h+1)}(l_h, \lambda-\rho)$. We have $\tau_{k-1} < \tau_k$ and $\lim_{k\to\infty} \tau_k = t_1$. From $c_k =$

$\Theta_n(\tau_k,\nu_k)=\frac{-\tau_{k-1}}{1+\tau_{k-1}\nu_{k-1}}$, we find $c_k<c_{k-1}$ and

$$\lim_{k\to\infty} c_k=\frac{-1}{\frac{1}{t_1}+\lambda-\rho}=\frac{-1}{\frac{1+(\lambda-\rho)\mathcal{H}_q}{-\mathcal{H}_q}+\lambda-\rho}=\mathcal{H}_q.$$

4.2. Odd case for $\alpha=\frac{\rho}{\lambda}$. Hitoshi Nakada recently observed that the dynamical systems $(\Omega_{1/2},\mu_{1/2},\mathcal{T}_{1/2})$ and $(\Omega_{\rho/\lambda},\mu_{\rho/\lambda},\mathcal{T}_{\rho/\lambda})$ are metrically isomorphic via M given in (III.49). In Section 3.2 we used this isomorphism to derive results for $(\Omega_{1/\lambda},\mu_{1/\lambda},\mathcal{T}_{1/\lambda})$ from $(\Omega_{1/2},\mu_{1/2},\mathcal{T}_{1/2})$ by applying Theorem III.51. For odd q and $\alpha=\frac{\rho}{\lambda}$ we can do the same and for this case the proof of Theorem III.62 is similar to the one for the even case with $\alpha=\frac{1}{\lambda}$ given in Section 3.2.

For $q=3$ this result (where one can take $K=1$) had been known for a long time for the nearest integer continued fraction expansion (the case $\alpha=1/2$) and for the singular continued fraction expansion ($\alpha=\frac{1}{2}(\sqrt{5}-1)$), cf. [**17**].

4.3. Odd case for $\alpha\in(\frac{\rho}{\lambda},1/\lambda]$. In this last case the natural extension Ω_α is given by $\Omega_\alpha=\bigcup_{n=1}^{4h+3}J_n\times[0,H_n]$. With intervals given by

$$\begin{aligned} J_{2n-1}&=[l_{n-1},r_n) \quad \text{for } n=1,2,\ldots,h+1\\ J_{2n}&=[r_n,l_n) \quad \text{for } n=1,2,\ldots,h \qquad \text{and} \qquad J_{2h+2}=[r_{h+1},r_0),\end{aligned}$$

and heights defined by

$$H_1=\frac{1}{\lambda+1},H_2=\frac{1}{\lambda} \qquad \text{and} \qquad H_n=\frac{1}{\lambda-H_{n-2}} \qquad \text{for } n=3,4,\ldots,2h+2.$$

In [**11**] was shown that $H_{2h}=\lambda-1,H_{2h+1}=\frac{\lambda}{2}$ and $H_{2h+2}=1$. If $\alpha=\frac{1}{\lambda}$ the intervals J_{2n-1} are empty, see Theorem III.60. Again we have $l_h=\frac{\alpha\lambda-1}{1-\alpha\lambda(\lambda-1)}$.

4.3.1. *Points in $\mathcal{D}^+$.* We saw in Section 4 that $\mathcal{D}^+\neq\emptyset$. The leftmost point of $\overline{\mathcal{D}^+}$ is given by (ρ,ρ). Using the same techniques as in the rest of this chapter yields

$$\begin{aligned} T_\alpha^{h+1}(\rho)&=(S^{-1}T)^hST(\rho)\\ &=B_{h+1}\begin{bmatrix}-1 & -1\\ 1 & \lambda-1\end{bmatrix}(-\rho)=\frac{\rho-1}{-\rho+\lambda-1}=\rho.\end{aligned}$$

It easily follows that $\mathcal{T}_\alpha^{h+1}(\rho,\rho)=(\rho,\rho)$.

Remark III.65. In [**8**] was shown that $\rho=[\overline{+1:1,(-1:1)^h}]$ from which immediately follows $T_\alpha^{h+1}(\rho)=\rho$. $\diamond$

We note that $\mathcal{T}_\alpha$ 'flips' $\mathcal{D}^+$ in the first step, in the sense that $\mathcal{T}_\alpha(\rho,\rho)$ is the rightmost point of $\mathcal{T}(\overline{D^+})$. The orientation is preserved in the next h steps, so we know that the rightmost point of $\mathcal{T}_\alpha^{h+1}(\overline{\mathcal{D}^+})$ is given by $\mathcal{T}_\alpha^{h+1}(\rho,\rho)=(\rho,\rho)$. Therefore we find that

$$\mathcal{T}_\alpha^{h+1}(\mathcal{D}^+)\cap\mathcal{D}^+=\emptyset.$$

So after one round all points in $\mathcal{D}^+$ are flushed out of $\mathcal{D}$. Furthermore, it is straightforward to check that $\mathcal{T}_\alpha(\mathcal{D}^+) \subset \mathcal{D}$, and that $\mathcal{T}_\alpha^i(\mathcal{D}^+) \cap \mathcal{T}_\alpha^j(\mathcal{D}^+) = \emptyset$ for $0 \le i < j \le h+1$.

4.3.2. *Points in $\mathcal{D}^-$.* As in the previous section we only describe the part of $\mathcal{D}^-$ on the right hand side of the line $t = -\delta_1$.

Lemma III.66. *If $\alpha \in \left(\frac{\rho}{\lambda}, \alpha_4\right]$, then $\mathcal{D}^-$ consists of two components: $\mathcal{D}_1$ bounded by the line $t = -\delta_1$ from the left, the line $v = H_{2h} = \lambda - 1$ from above and the graph of g from below and $\mathcal{D}_2$ bounded by the line $t = l_h$ from the left, the line $v = H_{2h+1} = \frac{\lambda}{2}$ from above and the graph of g from below.*

If $\alpha \in \left(\alpha_4, \frac{1}{\lambda}\right]$, then $\mathcal{D}^- = \mathcal{D}_1$.

Proof. For all $\alpha \in \left(\frac{\rho}{\lambda}, \frac{1}{\lambda}\right]$ we have that $g(-\delta_1) < H_{2h} < g(l_h)$. Furthermore $g(l_h) < H_{2h+1}$ if and only if $\alpha < \alpha_4$. $\square$

Again we must distinguish between two subcases; also see Figure 11.

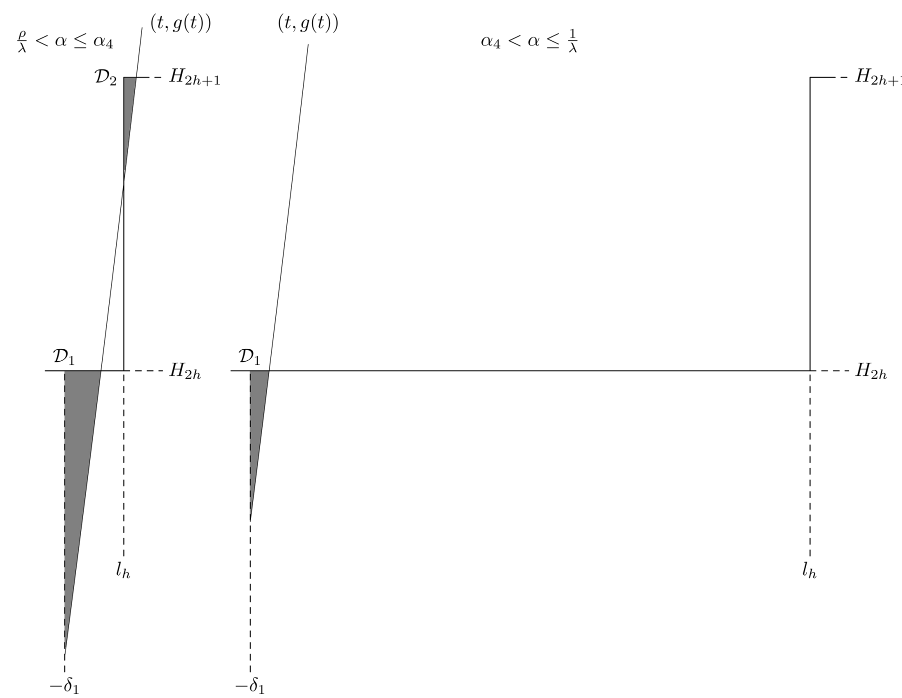


Figure 11. On the left we used $\alpha = 0.50098 < \alpha_4$, on the right we use $\alpha = 0.52 > \alpha_4$.

Proof of Theorem III.62 for $\frac{\rho}{\lambda} < \alpha \le \alpha_4$. First we note that $T^{h+1}(-\delta_1) = l_h$ and thus after one round all points in $\mathcal{D}_1$ are either flushed or send to $\mathcal{D}_2$.

We look at the orbit of l_h under T_α. We find $T_\alpha(l_h) = \frac{1+\alpha\lambda(\lambda+1)-2\lambda}{1-\alpha\lambda}$ and some calculations show that $r_1 < T_\alpha(l_h) < l_1$. We conclude that

$$\begin{aligned} T_\alpha^h(l_h) &= (S^{-1}T)^{h-1}S^{-2}T(l_h) = \begin{bmatrix} \lambda^2-\lambda+1 & \lambda-1 \\ -\lambda^3+\lambda^2+1 & -\lambda^2+\lambda+1 \end{bmatrix}(l_h) \\ &= \frac{2\lambda+(\alpha-1)\lambda^2-2}{((1-\alpha)\lambda^2+2\alpha+1-2\lambda)\lambda}. \end{aligned}$$

One can check that for $\alpha > \frac{\rho}{\lambda}$ we have $-\delta_1 < T_\alpha^h(l_h) < l_h$. We find

$$T_\alpha^{h+1}(l_h) = T_\alpha(T_\alpha^h(l_h)) = \frac{(\lambda^2-2\lambda-\alpha(\lambda^2+2)+3)\lambda}{2\lambda+(\alpha-1)\lambda^2-2} < r_1.$$

So we finally conclude that the image of l_h after one round of $2h+1$ steps is given by

$$\begin{aligned} T_\alpha^{2h+1}(l_h) &= (S^{-1}T)^h S^{-2}T(S^{-1}T)^{h-1}S^{-2}T(l_h) = \begin{bmatrix} \lambda^2+2 & \lambda \\ -2\lambda^2+\lambda & 2-2\lambda \end{bmatrix}(l_h) \\ &= \frac{(-\alpha+1)\lambda^2-2\alpha\lambda-\lambda+2}{(3\alpha-2)\lambda^2+(3-2\alpha)\lambda-2}. \end{aligned}$$

The right end point of $\mathcal{D}_2$ is given by $\left(\frac{-2\mathcal{H}_q}{\lambda\mathcal{H}_q+2}, \frac{\lambda}{2}\right)$. Since for $\frac{\rho}{\lambda} < \alpha \leq \alpha_4$ we have $\frac{-2\mathcal{H}_q}{\lambda\mathcal{H}_q+2} < \frac{(-\alpha+1)\lambda^2-2\alpha\lambda-\lambda+2}{(3\alpha-2)\lambda^2+(3-2\alpha)\lambda-2} < 0$, we find that all points are flushed out of $\mathcal{D}_2$ after one round. So all points in $\mathcal{D}_1$ are flushed after at most $3h+2$ steps.

Proof of Theorem III.62 for $\alpha_4 < \alpha \leq \frac{1}{\lambda}$. All points are flushed out of $\mathcal{D}_1$ after $h+1$ steps, since the line $t = -\delta_1$ is mapped by T_α to the line $t = l_0$. So after $h+1$ steps all points in $\mathcal{D}$ are mapped to points that lie on the right hand side of the line $t = l_h$ and thus outside of $\mathcal{D}_1$.

We need to check that $\mathcal{T}_\alpha^{h+1}(\mathcal{D}_1) \cap \mathcal{D}^+ = \emptyset$. The rightmost vertex of $\overline{\mathcal{D}_1}$ is given by $\left(\dfrac{-1}{\lambda-1+\frac{1}{\mathcal{H}_q}}, \lambda-1\right)$. With similar techniques as before we find that applying $(h+1)$-times $\mathcal{T}_\alpha$ to the rightmost vertex of $\mathcal{D}_1$ yields as first coordinate

$$\begin{aligned} &T_\alpha^{h+1}\left(\frac{-1}{\lambda-1+\frac{1}{\mathcal{H}_q}}\right) \\ &= \begin{bmatrix} -\lambda & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda^2-\lambda+1 & \lambda-1 \\ -\lambda^3+\lambda^2+1 & -\lambda^2+\lambda+1 \end{bmatrix}\left(\frac{-1}{\lambda-1+\frac{1}{\mathcal{H}_q}}\right) \\ &= \frac{1-2\mathcal{H}_q}{1+(\mathcal{H}_q-1)\lambda} < \rho. \end{aligned}$$

This completes the proof. ☐

5. Borel and Hurwitz constants for α-Rosen fractions

In this section we first prove the Borel-type Theorem III.6 and then derive a Hurwitz-type result for certain values of α.

5.1. Borel for α-Rosen fractions. Let $q \geq 3$ be an integer. Recall that for all $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$ and every G_q-irrational $x \in [(\alpha-1)\lambda, \alpha\lambda)$ the future t_n and past v_n satisfy

$$\mathcal{T}_\alpha^n(x,0) = (t_n, v_n) \quad \text{for all } n \geq 0. \tag{III.67}$$

We denote by r the number of steps in a round, so $r = p-1$ for even q and $r = 2h+1$ for odd q. Furthermore, for even q we define $\rho = 1$. Compare the following with Lemma 12 of [**30**].

Lemma III.68. *Let either q be even and $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$ or q be odd and $\alpha \in \left[\frac{1}{2}, \frac{\rho}{\lambda}\right]$. Let $\mathcal{F}$ denote the fixed point set in $\mathcal{D}$ of $\mathcal{T}_\alpha^r$. Then*

(*i*) $\mathcal{F} = \left\{ \mathcal{T}_\alpha^i \left(\frac{-\rho}{1+\lambda\rho}, \lambda - \rho\right) \,\middle|\, i = 0, 1, \ldots, r-1 \right\}$;

(*ii*) *For every x and every $n \geq 0$, $(t_n, v_n) \notin \mathcal{F}$;*

(*iii*) *For every G_q-irrational number x there are infinitely many n for which $(t_n, v_n) \notin \mathcal{D}$;*

(*iv*) *For each $i = 0, 1, \ldots, r-1$, let $x_i = T_\alpha^i \left(\frac{-\rho}{1+\lambda\rho}\right)$. Then for all $n \geq 0$, $\mathcal{T}_\alpha^n(x_i, 0) \notin \mathcal{D}$. However, $\mathcal{T}_\alpha^{kr}(x_i, 0)$ converges from below on the vertical line $x = x_i$ to ${\mathcal{T}_\alpha}^i \left(\frac{-\rho}{1+\lambda\rho}, \lambda - \rho\right)$.*

Proof. Assume that q is even and $\alpha \in \left(\frac{1}{2}, \frac{1}{\lambda}\right)$, the other cases can be proven in a similar way. In this case $r = p-1$ and we have by Corollary III.40 that $\left(\frac{-1}{\lambda+1}, \lambda - 1\right)$ is a fixed point of $\mathcal{T}_\alpha^{p-1}$. It follows that points in the $\mathcal{T}_\alpha$-orbit of $\left(\frac{-1}{\lambda+1}, \lambda - 1\right)$ must also be fixed points of $\mathcal{T}_\alpha^{p-1}$, which proves (*i*). In each fixed point of $\mathcal{T}_\alpha^{p-1}$ both of the coordinates have a periodic infinite expansion. For (*ii*) we note that by definition v_n has a finite expansion of length n and therefore for every x and every n we have $v_n \notin \mathcal{F}$. We conclude from the section on flushing on page 52, that for every G_q-irrational number x there are infinitely many n for which $(t_n, v_n) \notin \mathcal{D}$, which is (*iii*). Finally, for each $i = 0, 1, \ldots, r-1$ and every $n \geq 0$ the points $\mathcal{T}_\alpha^n(x_i, 0)$ are below the graph of f. The fixed points ${\mathcal{T}_\alpha}^i \left(\frac{-1}{1+\lambda}, \lambda - 1\right)$ are attractors for these points, cf. the proof of Lemma III.43. □

Proof of Theorem III.6. Let $q \geq 3$ be an integer and let x be a G_q-irrational. We first assume that we do not have a finite spectrum, so if q is even, we consider all $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$ and if q is odd we assume $\alpha \in \left[\frac{1}{2}, \frac{\rho}{\lambda}\right]$.

From Lemma III.68 (ii) we see that (t_n, v_n) can never be a fixed point for any $n \geq 0$. From (iii) we know that here are infinitely many n for which $(t_n, v_n) \notin \mathcal{D}$, so there are infinitely many $n \in \mathbb{N}$ for which

$$q_n^2 \left| x - \frac{p_n}{q_n} \right| \leq \mathcal{H}_q.$$

It remains to show that also in this case $\mathcal{H}_q$ cannot be replaced by a smaller constant. Take x such that $t_1 = \frac{-\rho}{1+\lambda\rho}$. By definition of Ω_α we know that $v_1 \leq \lambda - 1$ and since v_1 has a finite expansion we find $v_1 < \lambda - 1$. For all $l \geq 1$ we have $(t_{1+r\,l}, v_{1+r\,l}) \notin \mathcal{D}$ and for every $0 \leq i < r$ one has $\lim_{l\to\infty} \mathcal{T}_\alpha^i(t_{1+r\,l}, v_{1+r\,l}) = \mathcal{T}_\alpha^i \left(\frac{-\rho}{1+\lambda\rho}, \lambda - \rho \right)$. So, $\mathcal{H}_q$ cannot be replaced by a smaller constant.

Finally, assume q is odd and $\alpha \in \left(\frac{\rho}{\lambda}, \frac{1}{\lambda}\right]$. From the finite spectrum in Theorem III.62 it immediately follows that in this case there are infinitely many n for which

$$q_n^2 \left| x - \frac{p_n}{q_n} \right| \leq \mathcal{H}_q.$$

It remains to show that in this case the constant $\mathcal{H}_q$ cannot be replaced by a smaller constant. Consider (ρ, ρ), the fixed point of $\mathcal{T}_\alpha^{h+1}$. We find

$$\mathcal{T}_\alpha(\rho, \rho) = \left(\frac{1}{\rho} - \lambda, \frac{1}{\lambda + \rho} \right).$$

Note that $H_1 = \frac{1}{\lambda+1} < \frac{1}{\lambda+\rho} < \frac{1}{\lambda} = H_2$. Furthermore

$$f\left(\frac{1}{\rho} - \lambda\right) = \frac{\mathcal{H}_q}{1 - \mathcal{H}_q\left(\frac{1}{\rho} - \lambda\right)} = \frac{\rho}{\rho^2 + 1 - \rho\left(\frac{1}{\rho} - \lambda\right)} = \frac{1}{\lambda + \rho}.$$

So $\mathcal{T}_\alpha(\rho, \rho)$ lies on the graph of f bounding $\mathcal{D}$. Points in $\mathcal{D}$ on the left hand side of $x = -\delta_1$ are mapped into $\mathcal{D}$ by $\mathcal{T}_\alpha$ and we find $\mathcal{T}_\alpha^2(\rho, \rho), \mathcal{T}_\alpha^3(\rho, \rho), \ldots, \mathcal{T}_\alpha^{h-1}(\rho, \rho)$ all are in $\mathcal{D}$.

It is easy to check that $\mathcal{T}_\alpha\left(\frac{-1}{\lambda+\rho}, \lambda - \frac{1}{\rho}\right) = (\rho, \rho)$ and that $g\left(\frac{-1}{\lambda+\rho}\right) = \lambda - \frac{1}{\rho}$. We conclude that $\mathcal{T}_\alpha^h(\rho, \rho) = \mathcal{T}_\alpha^{-1}(\rho, \rho) = \left(\frac{-1}{\lambda+\rho}, \lambda - \frac{1}{\rho}\right)$ lies on the graph of g.

Now consider any point $(t_n, v_n) = (\rho, y)$, since ρ has a periodic infinite expansion we know $y \neq \rho$. However the periodic orbit of (ρ, ρ) is an attractor of the orbit of (ρ, y), so $\lim_{k\to\infty} \mathcal{T}_\alpha^{k(h+1)}(\rho, y) = (\rho, \rho)$. It follows that the constant $\mathcal{H}_q$ is best possible in this case. □

5.2. Hurwitz for α-Rosen fractions. For odd q and some values of α we can generalize Theorem III.6 to a Hurwitz-type theorem, which is the Haas-Series result mentioned in Section 1.1. From [**4**] it follows that for all $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$, for all $z \geq 0$ and for almost all x, the limit

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n | \Theta_j(x) \leq z\}$$

exists and equals the distribution function F_α, which satisfies

$$F_\alpha(z) = \bar{\mu}\left(\left\{ (t, v) \in \Omega_\alpha \,|\, v \leq f(t) = \frac{z}{1 - zt} \right\} \right),$$

where $\bar{\mu}$ is the invariant measure for $\mathcal{T}_\alpha$ given in [**11**]. Defining the Lenstra constant $\mathcal{L}_\alpha$ by

$$\mathcal{L}_a = \max\left\{ c > 0 \,\middle|\, \left(t, \frac{c}{1-ct}\right) \in \Omega_\alpha \text{ , for all } t \in [l_0, r_0] \right\}. \tag{III.69}$$

As mentioned in Section 1.2, the distribution function F_α is a linear map with positive slope for $z \in [0, \mathcal{L}_\alpha]$. Nakada showed in [**42**] that the Lenstra constant is equal to the Legendre constant whenever the latter constant exists. In his article he particularly mentioned Rosen fractions and α-expansions, but this result also holds for α-Rosen fractions. So if p/q is a G_q-rational and

$$q^2 \left| x - \frac{p}{q} \right| < \mathcal{L}_\alpha,$$

then p/q is an α-Rosen convergent of x.

Since for the standard Rosen fractions (where $\alpha = \frac{1}{2}$) one has that $\mathcal{L}_\alpha < \mathcal{H}_q$, the Haas-Series result does not follow from Theorem III.6, see also the discussion on the results of Legendre, Borel and Hurwitz in Section 1.2.

One wonders whether α-Rosen fractions could yield a continued fraction proof of the Haas-Series result for particular values of α. Proposition 4.3 of [**11**] states that for even α-Rosen-fractions

$$\mathcal{L}_a = \min\left\{ \frac{\lambda}{\lambda+2}, \frac{\lambda(2-\alpha\lambda^2)}{4-\lambda^2} \right\}.$$

Since $\mathcal{L}_\alpha < \mathcal{H}_q = \frac{1}{2}$, we see that a direct continued fraction proof of a Hurwitz-result cannot be given in this case. In [**11**] the more involved formula for $\mathcal{L}_\alpha$ for odd α-Rosen fractions was not given. For odd q we have the following proposition.

Proposition III.70. *Let $q \geq 3$ be an odd integer and let $\alpha_L = \frac{\mathcal{H}_q}{\lambda(1-\mathcal{H}_q)}$. Then $\mathcal{L}_\alpha < \mathcal{H}_q$ for $\alpha \in [1/2, \alpha_L)$, while $\mathcal{L}_\alpha = \frac{\alpha\lambda}{\alpha\lambda+1} > \mathcal{H}_q$ for $\alpha \in [\alpha_L, 1/\lambda]$.*

Proof. For every $\alpha \in [1/2, \alpha_L)$ there is a $C < \mathcal{H}_q$ such that $\left(t, \frac{C}{1-Ct}\right) \notin \Omega_\alpha$. We only prove this for $\alpha = \rho/\lambda$. Consider the point $\left(\rho, \frac{C}{1-C\rho}\right)$. This point is in Ω_ρ if the y-coordinate is smaller than top height $\frac{\lambda}{2}$. We have

$$\frac{C}{1-C\rho} < \frac{\lambda}{2} \text{ if and only if } C < \frac{\lambda}{\lambda\rho+2}.$$

So by (III.69) we have that $\mathcal{L}_\alpha = \frac{\lambda}{\lambda\rho+2} < \mathcal{H}_q$.

Let $\alpha \in [\alpha_L, 1/\lambda]$. Consider the point $\left(r_1, \frac{C}{1-Cr_1}\right)$, this point is in Ω_α if the y-coordinate is smaller than $H_1 = \frac{1}{\lambda+1}$. Using $r_1 = \frac{1}{\alpha\lambda} - \lambda$ we find that $\frac{C}{1-Cr_1} \leq \frac{1}{\lambda+1}$ if and only if $C \leq \frac{\alpha\lambda}{\alpha\lambda+1}$. For $C = \frac{\alpha\lambda}{\alpha\lambda+1}$ we easily find that all points $\left(t, \frac{C}{1-Ct}\right)$ with $t \in (l_0, r_0)$ are in Ω_α, so $\mathcal{L}_\alpha = \frac{\alpha\lambda}{\alpha\lambda+1}$. Finally we see that $\mathcal{L}_\alpha > \mathcal{H}_q$ if and only if $\alpha > \alpha_L$. $\square$

Thus the Hurwitz-type theorem of Haas-Series in the case of odd index q follows from Theorem III.6, Proposition III.70 and Nakada's result from [**42**] by using any $\alpha \in [\alpha_L, 1/\lambda]$.

IV
Quilting natural extensions for α-Rosen continued fractions

In this chapter we give a method that starts from the explicit domain of the natural extension of a Rosen fraction and determines the domains for the natural extensions for various α-Rosen fractions. One advantage of this approach is that one easily sees that these α-Rosen fraction maps determine isomorphic dynamical systems. In particular we find that the associated one-dimensional maps have the same entropy. This finding can be compared to results on the entropy of Nakada's α-continued fractions obtained by Nakada [**41**] and others [**39**], [**37**], [**44**].

1. Introduction

Let $q \in \mathbb{Z}, q \geq 3$ and $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$. For $\alpha \in \left[0, \frac{1}{\lambda}\right]$, we let $\mathbb{I}_\alpha := [\,(\alpha-1)\lambda, \alpha\lambda\,)$.

Recall from the Chapters I and III that the α-Rosen fraction operator for $x \in \mathbb{I}_\alpha$ is defined as

$$T_\alpha(x) = \frac{\varepsilon}{x} - \lambda \left\lfloor \frac{\varepsilon}{\lambda x} + 1 - \alpha \right\rfloor \text{ if } x \neq 0 \text{ and } T_\alpha(0) := 0. \tag{IV.1}$$

Setting $\alpha = 1/2$ yields the Rosen fractions.

The domain of the natural extension of T_α is denoted by Ω_α and refers to the largest region on which $\mathcal{T}_\alpha$ is bijective, where the two-dimensional operator $\mathcal{T}_\alpha$ is defined by

$$\mathcal{T}_\alpha(x,y) = \left(T_\alpha(x), \frac{1}{d(x)\lambda + \varepsilon(x)y}\right), \quad \text{for}(x,y) \in \Omega_\alpha. \tag{IV.2}$$

In [**11**] Dajani, Kraaikamp and Steiner used direct methods, similar to those of [**8**] for the classical Rosen fractions, to determine natural extensions for $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$.

In [**11**] it was shown that the domain of the natural extension is connected for all $\alpha \in [\,1/2, 1/\lambda\,]$. In this chapter we determine the least α_0 such that for all $\alpha \in (\,\alpha_0, 1/\lambda\,]$ the natural extension is connected. We prove Theorem I.47 announced in the introduction:

Theorem IV.3. *Fix $q \in \mathbb{Z}, q \geq 4$ and $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$ and let*

$$\alpha_0 = \begin{cases} \dfrac{\lambda^2 - 4 + \sqrt{\lambda^4 - 4\lambda^2 + 16}}{2\lambda^2} & \text{if } q \text{ is even,} \\ \\ \dfrac{\lambda - 2 + \sqrt{2\lambda^2 - 4\lambda + 4}}{\lambda^2} & \text{otherwise.} \end{cases}$$

(i) *Then $(\alpha_0, 1/\lambda]$ is the largest interval containing $1/2$ for which each domain of the natural extension of T_α is connected.*

(ii) *Furthermore, let*

$$\omega_0 = \begin{cases} 1/\lambda & \text{if } q \text{ is even,} \\ \dfrac{\lambda - 2 + \sqrt{\lambda^2 - 4\lambda + 8}}{2\lambda} & \text{otherwise.} \end{cases}$$

Then the entropy of the α-Rosen map for each $\alpha \in (\alpha_0, \omega_0]$ is equal to the entropy of the standard Rosen map. Here ω_0 cannot be replaced by a larger number.

Remark IV.4. For every q we have $\alpha_0 < \frac{1}{2} < \omega_0 \leq \frac{1}{\lambda}$; see Figure 1. ◇

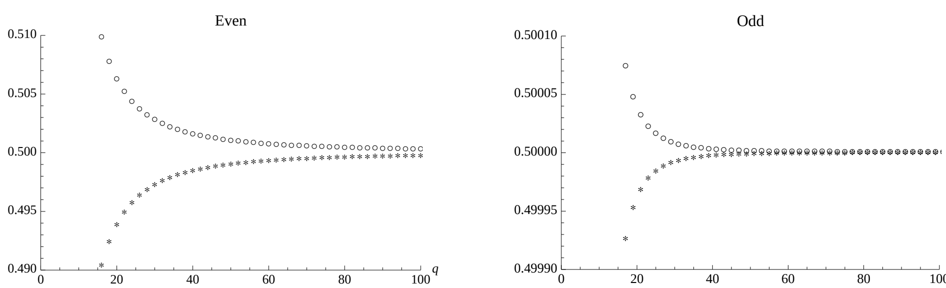


FIGURE 1. Some values of $\alpha_0(\circ)$ and of $\omega_0(*)$ with even index q on the left and odd index q on the right.

The value of the entropy of the standard Rosen map was found by H. Nakada [**42**] to be

$$\mathcal{C} \cdot \frac{(q-2)\pi^2}{2q} \, ,$$

where $\mathcal{C}$ is the normalizing constant (which depends on the parity of the index q) found in [**8**]. We give the value of $\mathcal{C}$ in (IV.23) for q even and in (IV.29) for q odd.

Our approach is to determine the domain of the natural extension from the domain for Rosen fractions, denoted here by $\Omega_{1/2}$. The shape of $\Omega_{1/2}$ was determined in [**8**]. It depends on the parity of q. We give the explicit formulas for $\Omega_{1/2}$ in Definitions IV.22 and IV.28 in the sections about the even and odd case, respectively.

For fixed λ and given α, we determine the domain by adding and deleting various regions to and from $\Omega_{1/2}$ by a process that we informally refer to as "quilting"; see Figure 2 for an example. The regions are determined from appropriate orbits (under the two-dimensional operator $\mathcal{T}_\alpha$) of the strips in $\Omega_{1/2}$ above intervals where the

T_α-"digits" (see below) differ from the $T_{1/2}$-digits. These strips are mapped by $\mathcal{T}_{1/2}$ to a region that must be deleted, their images under $\mathcal{T}_\alpha$ must be added; thereafter their further $\mathcal{T}_\alpha$-orbits are deleted and added, respectively. Our approach succeeds without great difficulty, because in a small number of steps the orbits of the added regions agree with the orbits of the deleted regions — the infinitely many potential holes are quilted over by the added regions.

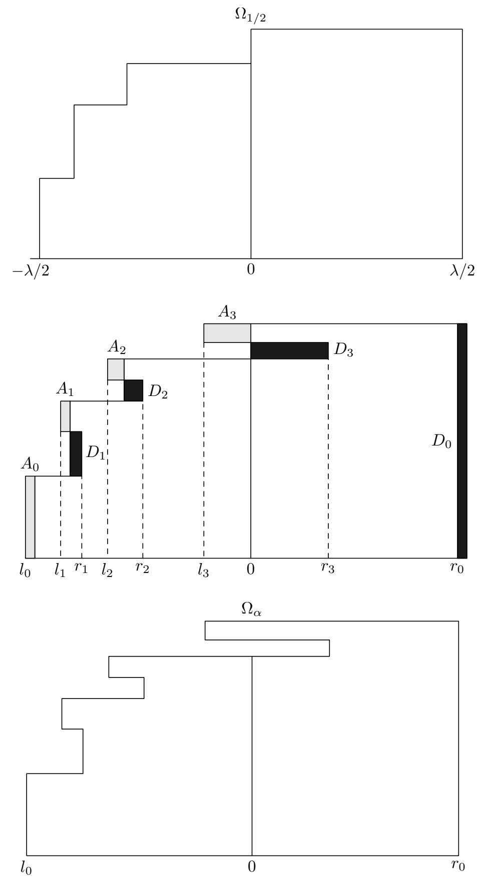


FIGURE 2. An example for $q = 8$. The first picture is $\Omega_{1/2}$ and below it is $\Omega_{1/2}$ with the blocks we add (A_0, A_1, A_2 and A_3) and the blocks we delete (D_0, D_1, D_2 and D_3). Below is the resulting Ω_α.

The next two subsections complete this Introduction by giving basic notation and then defining our building blocks, the basic deleted and added regions. In Section 2 we sketch the main argument of our approach — when the orbits of these basic regions agree after the same number of steps, entropy is preserved. We give an example of our techniques in Section 3 by re-establishing known results for certain classical Nakada α-fractions. In Sections 4 and 5 we give the proof of Theorem IV.3, in the even and odd index case, respectively. Finally, in Section 6 we indicate how our results can be extended to show that in the odd index case, the entropy of T_α decreases when $\alpha > \omega_0$.

1.1. Basic Notation. As in the previous chapter, for $x \in \mathbb{I}_\alpha$ we put $d(x) := d_\alpha(x) = \left\lfloor \left|\frac{1}{\lambda x}\right| + 1 - \alpha \right\rfloor$ and $\varepsilon(x) := \mathrm{sgn(x)}$. Again for $n \geq 1$ with $T_\alpha^{n-1}(x) \neq 0$ we put

$$\varepsilon_n(x) = \varepsilon_n = \varepsilon(T_\alpha^{n-1}(x)) \text{ and } d_n(x) = d_n = d(T_\alpha^{n-1}(x)).$$

When studying the dynamics of these maps, the orbits of the interval endpoints of $\mathbb{I}_\alpha = [\,(\alpha-1)\lambda, \alpha\lambda\,)$ are of utmost importance. Like in the previous chapter we use

$$\begin{aligned} l_0 &= (\alpha-1)\lambda \quad &&\text{and} \quad l_n = T_\alpha^n(l_0), \text{ for } n \geq 1\,, \\ r_0 &= \alpha\lambda &&\text{and} \quad r_n = T_\alpha^n(r_0), \text{ for } n \geq 1\,. \end{aligned}$$

For the orbit of the left end point in the Rosen case with $\alpha = \frac{1}{2}$ we define $\varphi_j = T_{1/2}^j(-\lambda/2)$ for integers $j \geq 0$.

The *cylinders* $\Delta(\varepsilon : d)$ are intervals in $\mathbb{I}_\alpha$ where $d(x)$ and $\varepsilon(x)$ are constant. For $x \in \mathbb{I}_\alpha$ they are defined by

$$\Delta(\varepsilon : d) = \{\, x \mid \mathrm{sgn}(x) = \varepsilon \,\mathrm{and}\, d(x) = d\,\}\,.$$

Letting

$$\delta_d := \delta_d(\alpha) = \frac{1}{(\alpha+d)\lambda}\,, \tag{IV.5}$$

we have

$$\Delta(+1:1) = (\,\delta_1, r_0\,) \quad \text{and} \quad \Delta(-1:1) = [\,l_0, -\delta_1\,]$$

For $d \geq 2$, we have *full* cylinders of the form

$$\Delta(+1:d) = (\,\delta_d, \delta_{d-1}\,] \text{ and } \Delta(-1:d) = [\,-\delta_{d-1}, -\delta_d\,).$$

Each full cylinder is mapped surjectively by T_α onto $\mathbb{I}_\alpha = [(\alpha-1)\lambda, \alpha\lambda)$; see [**11**].

For future reference, we introduce notation for *strips* that fiber over cylinders: for $(x, y) \in \Omega_\alpha$ we put

$$\mathcal{D}(\varepsilon : d) := \{\,(x, y) \mid x \in \Delta(\varepsilon : d)\,\}\,.$$

The strip $\mathcal{D}(\varepsilon : d)$ contains all points $(x, y) \in \Omega_\alpha$ with $\mathrm{sgn}(x) = \varepsilon$ and $d(x) = d$.

We use the *auxiliary sequence* B_n from (I.41) again:

$$B_0 = 0, \quad B_1 = 1, \quad B_n = \lambda B_{n-1} - B_{n-2}, \qquad \text{for } n = 2, 3, \ldots.$$

As remarked in the previous chapter it holds for $q = 2p$ with $p \geq 2$ an integer that

$$B_{p-1} = B_{p+1} = \frac{\lambda}{2} B_p \quad \text{and} \quad B_{p-2} = \left(\frac{\lambda^2}{2} - 1\right) B_p. \tag{IV.6}$$

In the odd case with $q = 2h+3$ for $h \in \mathbb{N}$ we have

$$B_{h+1} = B_{h+2}, \quad B_h = (\lambda-1)B_{h+1} \quad \text{and} \quad B_{h-1} = \left(\lambda^2 - \lambda - 1\right) B_{h+1}. \tag{IV.7}$$

1.2. Regions of changed digits; basic deletion and addition regions. Fix q, hence λ, and choose some $\alpha \in [\,0, \frac{1}{\lambda}\,]$. For α in the range that we consider, our intention is to find Ω_α from $\Omega_{1/2}$ using the operator $\mathcal{T}_\alpha$. Of course, for many points $(x, y) \in \Omega_{1/2}$, we have $\mathcal{T}_\alpha(x, y) = \mathcal{T}_{1/2}(x, y)$. We focus on the points $(x, y) \in \Omega_{1/2}$ where $\mathcal{T}_\alpha(x, y)$ differs from $\mathcal{T}_{1/2}(x, y)$. For instance for x between $\delta_2(1/2) = \frac{2}{5\lambda}$ and $\delta_2(\alpha) = \frac{1}{(\alpha+2)\lambda}$ we have $d_{1/2}(x) = 2$, but $d_\alpha(x) = 3$.

Definition IV.8. The *region of changed digits* is

$$C = \{\, (x, y) \in \Omega_{1/2} \;\mid\; x \in \mathbb{I}_\alpha \cap \mathbb{I}_{1/2} \text{ and } d_\alpha(x) \neq d_{1/2}(x) \,\}\,.$$

$\diamond$

It turns out that the region C is a disjoint union of rectangles; in general, for each digit d, the subset of C determined by x whose digit has changed to $d_\alpha(x)$ consists of two components, one with x negative, the other with positive x values. See Figure 4 for a schematic representation of C in the classical $\lambda = 1$ setting.

We identify a region of $\Omega_{1/2}$ that obviously cannot be part of the new natural extension, as its marginal projection onto the x-axis is outside of $\mathbb{I}_\alpha$.

Definition IV.9. The basic *deleted region* is the $\mathcal{T}_{1/2}$-image of the region of changed digits:

$$D_0 := \mathcal{T}_{1/2}(\,C\,)\,.$$

$\diamond$

Lemma IV.10. *For $\alpha < 1/2$, the basic deleted region is*

$$D_0 = \{\, (x, y) \in \Omega_{1/2} \mid x \geq \alpha\lambda \,\}\,.$$

For $\alpha > 1/2$, the basic deleted region is

$$D_0 = \{\, (x, y) \in \Omega_{1/2} \mid x < (\alpha - 1)\lambda \,\}\,.$$

Proof. The projection of C to the real line is a disjoint union of intervals. For $x < 0$ these intervals have the form $(-\delta_d(\alpha), -\delta_d(1/2)]$ and for $x > 0$ they have the form $[\delta_d(1/2), \delta_d(\alpha))$. The boundaries of the cylinders for the map T_z are determined by the values $\delta_d(z) = \frac{1}{\lambda(d+z)}$; thus, each of the components of the projection of C is at an end of a cylinder for $T_{1/2}$. Since for fixed d the function $z \mapsto \frac{1}{\lambda(d+z)}$ is decreasing, each component is at the right end of a cylinder for negative x, and at the left end of a cylinder for positive x values. Recall that these cylinders are mapped surjectively onto $\left[\frac{-\lambda}{2}, \frac{\lambda}{2}\right)$ and that on a cylinder $T_{1/2}$ itself is an increasing function for negative x and a decreasing function for positive x. Thus, this projection of C is exactly the $T_{1/2}$-preimage of $[\,\alpha\lambda, \lambda/2\,)$. From this, it easily follows that $\mathcal{T}_{1/2}(\,C\,) = \{\, (x, y) \in \Omega_{1/2} \mid x \geq \alpha\lambda \,\}$. The proof for $\alpha > \frac{1}{2}$ is similar. $\square$

Analogously to Definition IV.9, we define the *basic added region* to be

$$A_0 := \mathcal{T}_\alpha(\,C\,)\,.$$

Definition IV.11. Let D_0 and C be as defined above. Then

$$\Omega^*_\alpha := \left(\Omega_{1/2} \setminus \cup_{k=0}^{\infty} \mathcal{T}_\alpha^k(\, D_0\,) \right) \bigcup \cup_{k=1}^{\infty} \mathcal{T}_\alpha^k(C).$$

$\diamond$

2. Successful quilting results in equal entropy

Let μ denote the probability measure on Ω^*_α induced by $d\mu = \dfrac{dx\,dy}{(1+xy)^2}$. Let $\overline{\mathcal{B}}_\alpha$ denote the Borel σ-algebra of Ω^*_α.

Proposition IV.12. *Fix $q \in \mathbb{Z}, q \geq 3$ and $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, and choose some $\alpha \in [\,0, 1/\lambda\,]$. Let A_0 and D_0 be defined as above. Suppose that there is some natural number k such that*

$$\mathcal{T}_\alpha^k(A_0) = \mathcal{T}_\alpha^k(D_0)\,,$$

and $T_\alpha^i(D_0) \cap T_\alpha^j(D_0) = \emptyset$ for all $i, j < k, i \neq j$. Then $(\mathcal{T}_\alpha, \Omega^*_\alpha, \overline{\mathcal{B}}_\alpha, \mu\,)$ is isomorphic to $(\mathcal{T}_{1/2}, \Omega_{1/2}, \overline{\mathcal{B}}_{1/2}, \mu\,)$.

Proof. By [**8**], $\mathcal{T}_{1/2}$ is bijective (up to μ-measure zero) on $\Omega_{1/2}$. By the imposed condition $\mathcal{T}_\alpha$ is injective onto $\cup_{j=0}^{k-1} \mathcal{T}_\alpha^j(\,D_0\,)$. Therefore, we can define $f : \Omega_{1/2} \to \Omega_\alpha$ by

$$f(x,y) = \begin{cases} (x,y), & \text{if } (x,y) \in \Omega_{1/2} \setminus \bigcup_{j=0}^{k=1} T_\alpha^j(\,D_0\,), \\ \mathcal{T}_\alpha^{j+1} \circ \mathcal{T}_{1/2}^{-1} \circ \mathcal{T}_\alpha^{-j}(x,y), & \text{if } (x,y) \in \mathcal{T}_\alpha^j(\,D_0\,) \text{ for } j \in \{0, 1, \ldots k-1\}. \end{cases}$$

Since each of $\mathcal{T}_\alpha$ and $\mathcal{T}_{1/2}$ preserves the measure μ, one easily shows that this is an isomorphism. $\square$

Proposition IV.13. *With notation as above, let μ_α denote the marginal measure obtained by integrating μ on the fibers of $\pi : \Omega^*_\alpha \to \mathbb{I}_\alpha$. Further let $\mathcal{B}_\alpha$ denote the Borel σ-algebra of $\mathbb{I}_\alpha$. Then $(\mathcal{T}_\alpha, \Omega^*_\alpha, \overline{\mathcal{B}}_\alpha, \mu\,)$ is the natural extension of $(\,T_\alpha, \mathbb{I}_\alpha, \mathcal{B}_\alpha, \mu_\alpha\,)$.*

This can be proven with the variant of Schweiger's formalism (see Section 22 of [**52**]) that was used in [**11**].

We conclude that we will have proven part (ii) of Theorem IV.3 once we show that for each $\alpha \in (\,\alpha_0, \omega_0\,]$ there exists some k satisfying the hypotheses of Proposition IV.12.

From the above we conclude that if there exists a k as in Proposition IV.12, then $\Omega^*_\alpha = \Omega_\alpha$, where Ω_α is the domain for the natural extension of T_α. For $\alpha \in (\alpha_0, 1/2)$ we show in Lemma IV.26 that for $q = 2p$, with $p \geq 2$ a positive integer we have $k = p$ and in Lemma IV.32 that for $q = 2h+3$, with h a positive integer, we have $k = 2h+2$. Therefore, from now on we denote Ω^*_α by Ω_α.

Corollary IV.14. *Under the hypotheses of Proposition IV.12, the systems $(\,T_\alpha, \mathbb{I}_\alpha, \mathcal{B}_\alpha, \mu_\alpha\,)$ have the same entropy.*

PROOF. It is known that a system and its natural extension have the same entropy [**49**]. Since the natural extensions here are all isomorphic, they certainly have the same entropy. □

Remark IV.15. Rohlin [**49**] introduced the notion of natural extension explicitly in order to treat entropy. ◇

3. Classical case, $\lambda = 1$: Nakada's α-continued fractions

In this section we illustrate how our method works by deriving the form of the natural extension for Nakada's α-continued fraction (where $\lambda = 1$) from $\Omega_{1/2}$ in case $\sqrt{2} - 1 \leq \alpha \leq 1/2$.

In this case $\Omega_{1/2}$ is given by $[-1/2, 0) \times [0, g^2] \bigcup [0, 1/2) \times [0, g]$, where $g = \frac{\sqrt{5}-1}{2}$; see Figure 3.

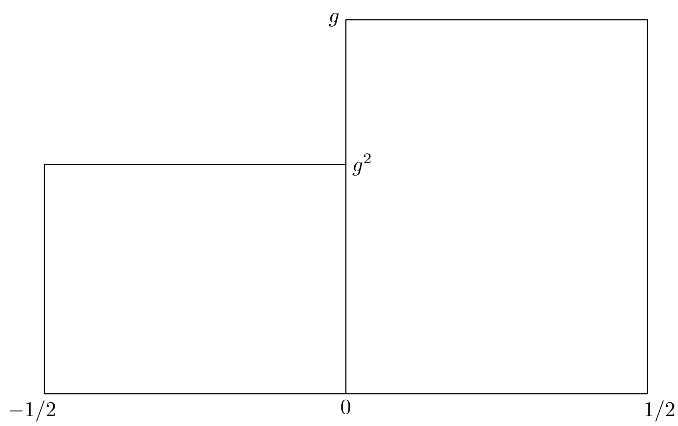


FIGURE 3. The natural extension $\Omega_{1/2}$ for $\lambda = 1$.

Our goal is to re-establish the following result from [**41**].

Theorem IV.16. *For all $\alpha \in [\sqrt{2} - 1, 1/2]$, the system $(\mathcal{T}_\alpha, \Omega_\alpha, \overline{\mathcal{B}}_\alpha, \mu)$ is isomorphic to $(\mathcal{T}_{1/2}, \Omega_{1/2}, \overline{\mathcal{B}}_{1/2}, \mu)$, where*

$$\begin{aligned}\Omega_\alpha := {} & [\,\alpha - 1, l_1\,) \times [\,0, g^2\,] \;\bigcup \\ & [\,l_1, r_1\,) \times \Big(\,[\,0, g^2\,] \cup [\,1/2, g\,]\Big) \;\bigcup \\ & [\,r_1, \alpha\,) \times [\,0, g\,]\,,\end{aligned}$$

and $\mathcal{B}_\alpha$ denotes the σ-algebra of μ-Borel subsets of Ω_α. Furthermore, the entropy of T_α with respect to the marginal measure of the above system equals $\dfrac{1}{\log(1+g)}\,\dfrac{\pi^2}{6}$.

Remark IV.17. The constancy of the entropy in this setting was first established by Moussa, Cassa and Marmi [**39**], the value at $\alpha = 1/2$ having been determined in [**41**]. We also note that Nakada and Natsui [**44**] explicitly show the isomorphism of these natural extensions (see their Appendix). ◇

3.1. Explicit form of the basic deleted and basis addition region. From Lemma IV.10 we find that the basic deleted region D_0 is given by $[\alpha, 1/2) \times [0, g)$, also see Figure 4.

Before we derive the shape of the basic added region A_0 we note that for d fixed, δ_d as defined in (IV.5) is a decreasing function in α. Note also that for $\alpha \in (\sqrt{2}-1, 1/2)$ one has $\delta_2(\alpha) < \alpha$.

Lemma IV.18. *The basic added region is given by*

$$A_0 = [\,\alpha - 1, -1/2) \times [0, g^2]\,.$$

Proof. Similarly to the proof of Lemma IV.10, we note that for $x < 0$ each component of the marginal projection of C is at the left end of a T_α-cylinder (see Figure 4) and is sent by (the locally increasing function) T_α to $[\,\alpha-1, -1/2)$. When $x > 0$, the component is at the right end of its cylinder and is also sent by T_α to $[\,\alpha - 1, -1/2)$. One trivially checks that no other points of $\mathbb{I}_\alpha$ are sent to this subinterval.

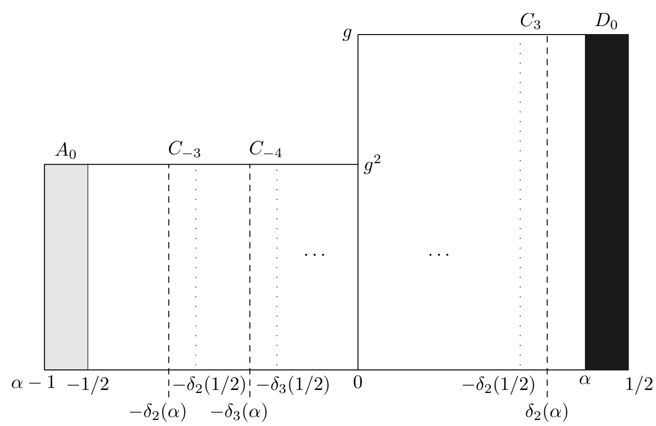


Figure 4. Regions of change of digit, basic deletion D_0 and basic addition A_0. To aid visualization, we use here $C_{\varepsilon d} = C \cap \Delta(\varepsilon : d)$. Note that $C = \bigcup_{\{\varepsilon,d\}} C_{\varepsilon,d}$.

We now discuss the y-coordinates of points in $\mathcal{T}_\alpha(A_0)$. The fibers of $\Omega_{1/2}$ above cylinders where $x < 0$ are of the form $[0, g^2]$, while above cylinders with $x > 0$ the fibers are of the form $[0, g]$. Now, for $(x, y) \in \mathcal{D}(\varepsilon : d)$, the y-coordinate of $\mathcal{T}_\alpha(x, y)$ is $1/(d + \varepsilon y)$. Thus, $\mathcal{T}_\alpha$ sends $\mathcal{D}(-1 : d) = [-\delta_{d-1}, -\delta_d) \times [0, g^2]$ and $\mathcal{D}(+1 : d) = [\delta_d, \delta_{d-1}) \times [0, g]$ to horizontal strips whose y-values lie in $\left[\frac{1}{d+g}, \frac{1}{d-g^2}\right]$. But, $g^2 = 1 - g$ and thus this image is directly above that of the horizontal strip given by $\mathcal{T}_\alpha$ applied to $\mathcal{D}(-1 : d+1)$ and $\mathcal{D}(+1 : d+1)$. Now, the greatest y-value of $\mathcal{T}_\alpha(C)$ comes from the intersection of the projection of C with $\mathcal{D}(-1 : 3)$. Thus, $A_0 = [\,\alpha - 1, -1/2) \times [0, 1/(3 - g^2)\,)$. Since $1/(3 - g^2) = g^2$, the result follows. ☐

3.2. Quilting. We now show that the $\mathcal{T}_\alpha$-orbits of the added regions eventually match the orbits of the deleted regions, and thus $\mathcal{T}_\alpha$ is bijective (modulo μ-measure zero) on Ω_α .

Lemma IV.19. *The following equality holds:*

$$l_2 = r_2 \, .$$

Furthermore, there is a $d \in \mathbb{N}$ such that

$$l_1 \in \Delta(-1:d) \ \text{ and } \ r_1 \in \Delta(1:d-1) \, .$$

PROOF. We have that $l_0 = \alpha - 1 \in \Delta(-1:2)$ and $r_0 = \alpha \in \Delta(+1:2)$, giving

$$l_1 = \frac{2\alpha - 1}{1-\alpha} \ \text{ and } \ r_1 = \frac{1-2\alpha}{\alpha} \, .$$

Therefore,

$$l_2 = \frac{1-\alpha}{1-2\alpha} - d \ \text{ and } \ r_2 = \frac{\alpha}{1-2\alpha} - d' \, ,$$

with d, d' the appropriate T_α-digits. Now, $l_2 - r_2 = 1 + d' - d$ and is the difference of two elements in $\mathbb{I}_\alpha$, a half-open interval of length one. We thus conclude that $d' = d - 1$ and $l_2 = r_2$. $\square$

The orbit of the basic addition region, A_0 , is quickly synchronized with that of the basic deletion region, D_0 . Recall that $\mathcal{D}(\varepsilon : d)$ is the strip that fibers over the interval $\Delta(\varepsilon : d)$.

Lemma IV.20. *We have*

$$\mathcal{T}_\alpha^2(\, A_0\,) = \mathcal{T}_\alpha^2(\, D_0\,) \, .$$

PROOF. Let $A_1 := \mathcal{T}_\alpha(\, A_0\,)$. Since $A_0 \subset \mathcal{D}(-1:2)$, an elementary calculation shows that

$$A_1 = [\, l_1\,, 0) \times [\, 1/2, g\,] \, .$$

Similarly, defining $D_1 := \mathcal{T}_\alpha(\, D_0\,)$, one has $D_1 \subset \mathcal{D}(1:2)$, and finds

$$D_1 = (\, 0, r_1\,] \times [\, g^2, 1/2\,] \, ;$$

see Figure 5.

With d as in Lemma IV.19, let

$$A_1' := A_1 \cap \mathcal{D}(-1:d) \quad \text{and} \quad D_1' := D_1 \cap \mathcal{D}(+1:d-1) \, .$$

By elementary calculation, Lemma IV.19, and an application of the identity $g^2 = 1 - g$, one finds that

$$\mathcal{T}_\alpha(A_1') = \mathcal{T}_\alpha(D_1') = [\, l_2, \alpha) \times \left[\, \frac{2}{2d-1}, \frac{1}{d-g} \,\right] \, .$$

Each of $A_1 \setminus A_1'$ and $D_1 \setminus D_1'$ projects to the union of full cylinders:

$$A_1 \setminus A_1' = \bigcup_{m=d+1}^{\infty} \Delta(-1:m) \times [\, 1/2, g\,] \ \text{ and } \ D_1 \setminus D_1' = \bigcup_{m=d}^{\infty} \Delta(+1:m) \times [\, g^2, 1/2\,] \, .$$

From this and $\mathcal{T}_\alpha\big(\, \Delta(-1:m) \times [\, 1/2, g\,]\,\big) = \mathcal{T}_\alpha\big(\, \Delta(+1:m-1) \times [\, g^2, 1/2\,]\,\big)$, we conclude that $\mathcal{T}_\alpha(\, A_1\,) = \mathcal{T}_\alpha(\, D_1\,)$, and the result follows. $\square$

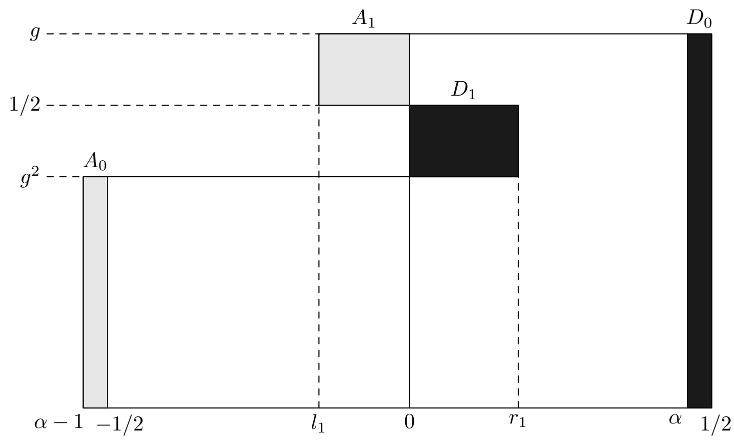


FIGURE 5. Representative region Ω_α for $\alpha \in (\sqrt{2}-1,\ 1/2)$; $q = 3$.

3.3. Isomorphic systems. In this subsection, we complete the proof of Theorem IV.16.

Corollary IV.21. *We have*

$$\begin{aligned}\Omega_\alpha = [\,\alpha-1, l_1\,) \times [\,0, g^2\,] \; &\bigcup \\ [\,l_1, r_1\,) \times \Big([\,0, g^2\,] \cup [\,1/2, g\,] \Big) \; &\bigcup \\ [\,r_1, \alpha\,) \times [\,0, g\,] \,. &\end{aligned}$$

Furthermore, the μ-area of Ω_α equals that of $\Omega_{1/2}$.

PROOF. From Corollary IV.11

$$\Omega_\alpha = \Big(\Omega_{1/2} \setminus (\,D_0 \cup D_1\,) \Big) \bigcup (A_0 \cup A_1) \;.$$

This proves the first part. By definition of A_0 and by Lemma IV.10, A_0 and D_0 are the images of the union of the change of digit regions under $\mathcal{T}_\alpha$ and $\mathcal{T}_{1/2}$, respectively. Since both $\mathcal{T}_\alpha$ and $\mathcal{T}_{1/2}$ are μ-measure preserving, $\mu(D_0) = \mu(A_0)$. Again since $\mathcal{T}_\alpha$ preserves measure, we also have $\mu(D_1) = \mu(A_1)$. $\square$

See Figure 5 for an example of the shape of Ω_α and the deleted and added regions.

Of course, the equality of the areas is already implied by the arguments of Section 2. From Lemma IV.20 and Proposition IV.12 we find that for $\lambda = 1$ and $\sqrt{2}-1 \le \alpha \le \frac{1}{2}$ the system $(\,T_\alpha, \mathbb{I}_\alpha, \mathcal{B}_\alpha, \mu_\alpha\,)$ is isomorphic to $(\,T_{1/2}, \mathbb{I}_{1/2}, \mathcal{B}_{1/2}, \mu_{1/2}\,)$. From Corollary IV.14 it follows that the entropy of the system $(\,T_\alpha, \mathbb{I}_\alpha, \mathcal{B}_\alpha, \mu_\alpha\,)$ is equal to that of the NICF, which is known to be $\dfrac{1}{\log(1+g)}\,\dfrac{\pi^2}{6}$. This finishes the proof of Theorem IV.16.

4. Even q ; $\alpha \in (\alpha_0, 1/2\,]$

The natural extension for the Rosen fractions was determined in [**8**]. The exact form of the domain depends on the parity of the index q. In this section we describe quilting for even indices q.

4.1. Natural extensions for Rosen fractions. The domain Ω_α was given in [**8**] as in the following definition; also see the beginning of Section 3.2. Recall that $\varphi_j = T^j_{1/2}(-\lambda/2)$.

Definition IV.22. Let $q = 2p$ for $p \in \mathbb{N}$ and $p \geq 2$. Let J_j be defined as follows:

$$J_j = [\varphi_{j-1}, \varphi_j) \quad \text{for} \quad j \in \{1, 2, \ldots, p-1\} \quad \text{and} \quad J_p = \left[0, \frac{\lambda}{2}\right).$$

Put $L_1 = 1/(\lambda + 1)$ and $L_j = 1/(\lambda - L_{j-1})$ for $j \in \{2, \ldots, p-1\}$. Further, set $K_j = [0, L_j]$ for $j \in \{1, 2, \ldots, p-1\}$ and $K_p = [0, 1]$. Then

$$\Omega_{1/2} = \bigcup_{j=1}^{p} J_j \times K_j \,.$$

◇

In [**8**] is shown that $L_{p-1} = \lambda - 1$. See Figure 2 for an example with $q = 8$.

The normalizing constant $\mathcal{C}$ such that $\mathcal{C} d\mu$ gives a probability measure on Ω_α is

$$\mathcal{C} = \frac{1}{\log\left((1 + \cos \pi/q\,)/\sin \pi/q\,\right)}\,, \tag{IV.23}$$

see Lemma 3.2 of [**8**].

The key to understanding the system for $\mathcal{T}_\alpha$ is that for all $q = 2p$, one has that $l_p = r_p$ for all of our α. In proving this, it is convenient to use the fact that the orbits of r_0 and $-r_0$ coincide after one application of T_α, so $T^j_\alpha(r_0) = T^j_\alpha(-r_0)$ for $j \geq 1$. We start from $-r_0$ instead of r_0.

Lemma IV.24. *For any $\alpha > \alpha_0$, the T_α-expansion of both l_0 and $-r_0$ starts as $[(-1:1)^{p-1}, \ldots]$.*

Proof. Like in the previous chapter we use Möbius transformations from Definition I.38. Recall that

$$S = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Any x with a negative sign and ones for its first $p-1$ digits is to the left of the appropriate $(p-2)$nd pre-image of $-\delta_1$. This pre-image is found by applying $p-2$ times $\mathcal{T}_\alpha^{-1}$ with $\varepsilon = -1$ and $d = 1$, so it is given by $(S^{-1}T)^{-p+2}(-\delta_1)$, cf. Lemma III.37.

From Lemma III.38 and (IV.6) we find

$$(S^{-1}T)^{-p+2} = \frac{B_p}{2} \begin{bmatrix} -\lambda^3 + 3\lambda & -\lambda^2 + 2 \\ \lambda^2 - 2 & \lambda \end{bmatrix}.$$

Thus the appropriate $(p-2)$nd pre-image of $-\delta_1$ is given by

$$T_\alpha^{-p+2}(-\delta_1) = \frac{(\lambda^3-3\lambda)\delta_1 - \lambda^2+2}{(-\lambda^2+2)\delta_1+\lambda} = -\frac{\alpha(\lambda^3-2\lambda)+\lambda}{\alpha\lambda^2+2}.$$

In particular, $-r_0$ starts with $p-1$ digits one if

$$-r_0 = -\alpha\lambda < -\frac{\alpha(\lambda^3-2\lambda)+\lambda}{\alpha\lambda^2+2}.$$

Rewriting this inequality yields that it holds whenever

$$\alpha > \alpha_0 := \frac{\lambda^2-4+\sqrt{\lambda^4-4\lambda^2+16}}{2\lambda^2}.$$

Finally, if $\alpha > \alpha_0$, then it immediately follows that l_0 also starts with $p-1$ ones since $l_0 = (\alpha-1)\lambda < -\alpha\lambda = -r_0$. □

Lemma IV.25. *For $\alpha \in (\alpha_0, 1/2)$, $r_p = l_p$. Furthermore, there is $d \in \mathbb{N}$ such that*

$$l_{p-1} \in \Delta(-1:d) \ \textit{ and } \ r_{p-1} \in \Delta(1:d-1)\ .$$

Proof. As in the proof of the previous lemma, we use Lemma III.38 and (IV.6) and find

$$\begin{aligned} l_{p-1} = T_\alpha^{p-1}(l_0) &= (S^{-1}T)^{p-1}\big(\,(\alpha-1)\lambda\,\big) \\ &= \begin{bmatrix} -2 & -\lambda \\ \lambda & \lambda^2-2 \end{bmatrix}\big(\,(\alpha-1)\lambda\big) \quad = \frac{(1-2\alpha)\lambda}{\alpha\lambda^2-2}\ , \end{aligned}$$

and similarly, since $T_\alpha(r_0) = T_\alpha(-r_0)$,

$$r_{p-1} = T_\alpha^{p-1}(-r_0) = \frac{(2\alpha-1)\lambda}{(1-\alpha)\lambda^2-2}\ .$$

For $\alpha \in (\alpha_0, \frac{1}{2})$ one has

$$\alpha\lambda^2-2<0, \quad (1-2\alpha)\lambda>0 \quad \text{and} \quad (1-\alpha)\lambda^2-2<0$$

and we find

$$\left|\frac{1}{l_{p-1}}\right| - \left|\frac{1}{r_{p-1}}\right| = \frac{\alpha\lambda^2-2}{(2\alpha-1)\lambda} - \frac{(1-\alpha)\lambda^2-2}{(2\alpha-1)\lambda} = \lambda\ .$$

But, then $T_\alpha(l_{p-1}) - T_\alpha(r_{p-1})$ is an integer multiple of λ. However, this is the difference of two elements of $\mathbb{I}_\alpha$, and thus this multiple must be zero. We conclude that $r_p = l_p$. That the digits of these points are as claimed follows as in the proof of Lemma IV.19. □

To describe the initial orbits of A_0 and D_0, we use the following sequence.

$$H_1 = \frac{1}{\lambda} \ \text{ and } \ H_i = \frac{1}{\lambda - H_{i-1}} \quad \text{for } i \geq 2\,.$$

In [**8**] it is shown that $H_{p-1} = \frac{\lambda}{2}$.

Lemma IV.26. *For $\alpha \in (\alpha_0, 1/2)$, $\mathcal{T}_\alpha^p(A_0) = \mathcal{T}_\alpha^p(D_0)$.*

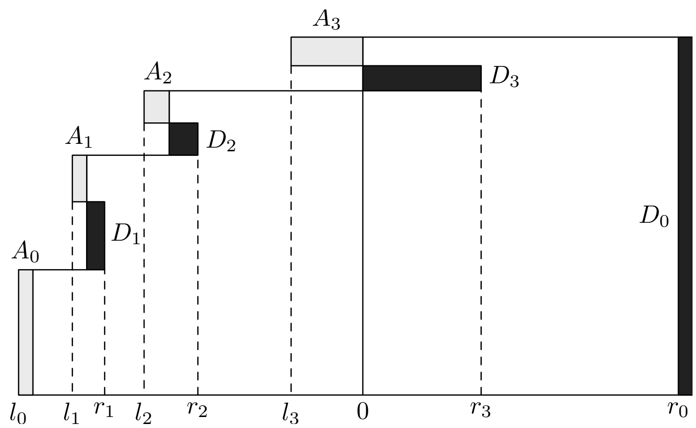


FIGURE 6. The added and deleted rectangles for $\alpha = 0.48$ in the natural extensions for $q = 8$. Here $A_i = \mathcal{T}^i(A_0)$ and $D_i = \mathcal{T}^i(D_0)$.

PROOF. One easily checks that the basic added region and basic deleted regions are

$$A_0 = [\,l_0, -\lambda/2\,) \times [0, L_1] \text{ and } D_0 = [r_0, \lambda/2\,) \times [\,0, 1]\,.$$

Since $l_0 < -\lambda/2 < -r_0$, we find from Lemma IV.24 that all $x \in [\,l_0, -\lambda/2\,)$ share the same first $p-1$ of their T_α-digits. Since the first $p-1$ of the $T_{1/2}$ digits of $-\lambda/2$ are all ones, we have that $T_\alpha^{p-1}(-\lambda/2) = T_{1/2}^{p-1}(-\lambda/2)$ and from [**8**] we know $\varphi_{p-1} = T_{1/2}^{p-1}(-\lambda/2) = 0$.

For the second coordinate we recall that $1/(\lambda - L_{p-1}) = 1$. We find that

$$\mathcal{T}_\alpha^{p-1}(A_0) = [\,l_{p-1}, 0) \times [\,H_{p-1}, 1\,]\,.$$

Paying attention to sign and orientation, one finds that

$$\mathcal{T}_\alpha(D_0) = [\,\varphi_1, r_1) \times [\,L_1, H_1\,)$$

and thus

$$\mathcal{T}_\alpha^{p-1}(D_0) = (\,0, r_{p-1}] \times [\,L_{p-1}, H_{p-1}\,]\,.$$

Analogously to the classical case, with d from Lemma IV.25, we let

$$A'_{p-1} := \mathcal{T}_\alpha^{p-1}(A_0) \cap \mathcal{D}(-1:d)\,,$$

and

$$D'_{p-1} := \mathcal{T}_\alpha^{p-1}(D_0) \cap \mathcal{D}(+1:d-1)\,.$$

We show that the $\mathcal{T}_\alpha$ images of A'_{p-1} and D'_{p-1} agree. From Lemma IV.25, $H_{p-1} = \frac{\lambda}{2}$ and $L_{p-1} = \lambda - 1$ we find

$$\begin{aligned}
\mathcal{T}_\alpha(A'_{p-1}) &= [l_p, r_0) \times \left[\frac{1}{d\lambda - H_{p-1}}, \frac{1}{d\lambda - 1}\right] \\
&= [l_p, r_0) \times \left[\frac{1}{(d-\frac{1}{2})\lambda}, \frac{1}{d\lambda - 1}\right], \\
\mathcal{T}_\alpha(D'_{p-1}) &= [r_p, r_0) \times \left[\frac{1}{(d-1)\lambda + H_{p-1}}, \frac{1}{(d-1)\lambda + L_{p-1}}\right] \\
&= [l_p, r_0) \times \left[\frac{1}{(d-\frac{1}{2})\lambda}, \frac{1}{d\lambda - 1}\right].
\end{aligned}$$

Analogously to the proof of Lemma IV.20 each of $A_1 \setminus A'_{p-1}$ and $D_1 \setminus D'_{p-1}$ projects to the union of full cylinders:

$$\begin{aligned}
A_1 \setminus A'_{p-1} &= \bigcup_{m=d+1}^{\infty} \Delta(-1:m) \times [\,H_{p-1}, 1\,] \text{ and} \\
D_1 \setminus D'_{p-1} &= \bigcup_{m=d}^{\infty} \Delta(+1:m) \times [\,L_{p-1}, H_{p-1}\,].
\end{aligned}$$

From this and $\mathcal{T}_\alpha\big(\Delta(-1:m) \times [\,H_{p-1}, 1\,]\big) = \mathcal{T}_\alpha\big(\Delta(+1:m-1) \times [\,L_{p-1}, H_{p-1}\,]\big)$, we conclude that $\mathcal{T}_\alpha(\,A_{p-1}\,) = \mathcal{T}_\alpha(\,D_{p-1}\,)$. The result follows. $\square$

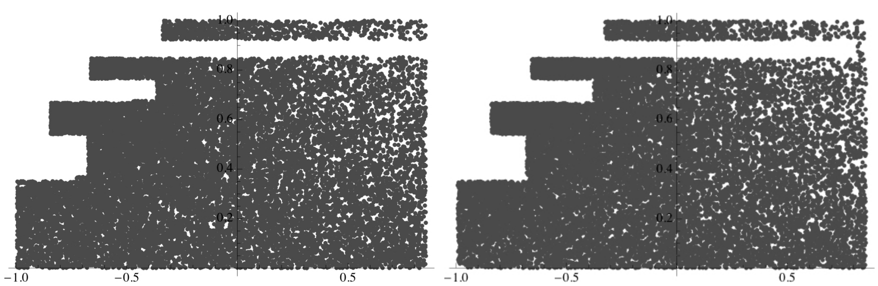


FIGURE 7. Change of topology at $\alpha = \alpha_0$: Simulations of the natural extension for $q = 8$ with on the left $\alpha = \alpha_0 - 0.001$ and on the right $\alpha = \alpha_0 + 0.001$.

Lemma IV.27. *The region Ω_{α_0} is not connected.*

PROOF. From the proof of Lemma IV.24 it follows that the first $p-2$ steps are similar for the case $\alpha = \alpha_0$, and A_j and D_j are as before for $j = 1, \ldots, p-2$. However we find that if $\alpha = \alpha_0$ then $r_{p-2} = T_\alpha^{p-2}(r_0) = -\delta_1$. Hence

$$D_{p-1} = \mathcal{T}_{\alpha_0}^{p-1}(D_0) = (\,0, \alpha_0\lambda] \times [\,L_{p-1}, H_{p-1}\,].$$

Furthermore,

$$A_{p-1} = \mathcal{T}_{\alpha_0}^{p-1}(A_0) = (\,l_{p-1}, 0] \times [\,H_{p-1}, 1\,].$$

Like in the proof of Lemma IV.26 the added blocks in all consecutive steps will be cancelled out by deleted blocks (and there will be some extra blocks deleted). Therefore, the entire rectangle $(0, \alpha_0\lambda] \times [L_{p-1}, H_{p-1}]$ will remain deleted and Ω_{α_0} is indeed disconnected. $\square$

We know from [**11**] that for $\alpha \in \left[\frac{1}{2}, \frac{1}{\lambda}\right]$ the region Ω_α is connected. Thus Lemma IV.26 and Lemma IV.27 prove part (i) of Theorem IV.3 for even index q. From Lemma IV.26, Proposition IV.12 and Corollary IV.11 it follows that for $\alpha \in \left(\alpha_0, \frac{1}{2}\right]$ and even q the entropy of the α-Rosen map equals that of the standard Rosen map. In Section 6 we finish the proof of part (ii) of Theorem IV.3.

5. Odd q; $\alpha \in (\alpha_0, 1/2]$

In this section we fix $q = 2h+3$ for a positive integer h and consider $\alpha \in (\alpha_0, 1/2]$. Let ρ be the positive root of $\rho^2 + (2-\lambda)\rho - 1 = 0$, also see (III.55).

5.1. Natural extensions for Rosen fractions. We start with the definition of $\Omega_{1/2}$ from [**8**], again using $\varphi_j = T^j_{1/2}(-\lambda/2)$.

Definition IV.28. Let $q = 2h+3$, for $h \geq 1$. Set

$$J_{2k} = [\varphi_{h+k}, \varphi_k), \text{ for } k \in \{1, \cdots, h\},$$
$$J_{2k+1} = [\varphi_k, \varphi_{h+k+1}), \text{ for } k \in \{0, 1, \cdots, h\},$$

and $J_{2h+2} = [0, \frac{\lambda}{2})$. Put

$$L_1 = \frac{1}{\lambda + 1/\rho}, \quad L_2 = \frac{1}{\lambda + \rho} \quad \text{and} \quad L_j = \frac{1}{\lambda - L_{j-2}} \quad \text{for} \quad 2 < j \leq 2h+2.$$

Let $K_j = [0, L_j]$ for $j \in \{1, \cdots, 2h+1\}$ and let $K_{2h+2} = [0, \rho]$.

Then

$$\Omega_{1/2} = \bigcup_{j=1}^{2h+2} J_j \times K_j.$$

$\diamond$

Furthermore, $L_{2h} = \lambda - \frac{1}{\rho}$ and $L_{2h+1} = \lambda - \rho$.

Also from [**8**] we have that the normalizing constant $\mathcal{C}$ such that $\mathcal{C}d\mu$ gives a probability measure on Ω_α is

$$\mathcal{C} = \frac{1}{\log(1+\rho)}. \tag{IV.29}$$

Remark IV.30. In [**8**] it is shown that $-\frac{2}{3\lambda} < \varphi_h < -\frac{2}{5\lambda}$, $\varphi_{2h+1} = 0$, and

$$-\frac{\lambda}{2} \leq \varphi_j < -\frac{2}{3\lambda} \quad \text{for} \quad j \in \{0, 1, \ldots, h-1\} \cup \{h+1, \ldots, 2h\}.$$

Note that $-\frac{2}{3\lambda} = -\delta_1\left(\frac{1}{2}\right)$ and $-\frac{2}{5\lambda} = -\delta_2\left(\frac{1}{2}\right)$. $\diamond$

Similarly to the even case, for each of our α, the basic added region and basic deleted regions are given by

$$A_0 = [\, l_0, -\lambda/2) \times [0, L_1]\,, \quad D_0 = [r_0, \lambda/2) \times [0, \rho]\,.$$

In this section we prove that for $q = 2h+3$ one has that $l_{2h+2} = r_{2h+2}$ for all $\alpha \in \left(\alpha_0, \frac{1}{2}\right)$ and consequently that the added blocks coincide with the deleted blocks after $2h+2$ steps. Here also, we use the fact that the orbits of r_0 and $-r_0$ coincide after one application of T_α.

Lemma IV.31. *For any $\alpha \in \left(\alpha_0, \frac{1}{2}\right]$, the T_α-expansion of both l_0 and $-r_0$ starts as*
$[\,(-1:1)^h, (-1:2), (-1:1)^h, \dots\,]$.

Proof. A point $x \in [l_0, r_0)$ has starting T_α digits $[\,(-1:1)^h, (-1:2), (-1:1)^h]$ if and only if $x < \left((S^{-1}T)^{h-1}S^{-2}T(S^{-1}T)^h\right)^{-1}(-\delta_1)$. We find

$$\begin{aligned}\left((S^{-1}T)^{h-1}S^{-2}T(S^{-1}T)^h\right)^{-1}(-\delta_1) &= \begin{bmatrix} (3-2\lambda)\lambda & 2(1-\lambda) \\ \lambda^2-2 & \lambda \end{bmatrix}(-\delta_1) \\ &= \frac{-2\alpha\lambda^2+2\alpha\lambda-\lambda}{\alpha\lambda^2+2}.\end{aligned}$$

We find that $-r_0 = -\alpha\lambda < \dfrac{-2\alpha\lambda^2+2\alpha\lambda-\lambda}{\alpha\lambda^2+2}$ if and only if $\alpha > \frac{\lambda-2+\sqrt{2\lambda^2-4\lambda+4}}{\lambda^2}$. Obviously, $l_0 < -r_0$ and we are done. $\square$

Lemma IV.32. *For $\alpha \in (\alpha_0, 1/2)$ one has that $\mathcal{T}^{2h+2}(A_0) = \mathcal{T}^{2h+2}(D_0)$.*

Proof. One has

$$\begin{aligned} l_{2h+1} = T^{2h+1}(l_0) = (S^{-1}T)^h S^{-2}T(S^{-1}T)^h(l_0) &= \begin{bmatrix} -2 & -\lambda \\ \lambda & 2\lambda-2 \end{bmatrix}((\alpha-1)\lambda) \\ &= \frac{(-2\alpha+1)\lambda}{(\alpha-1)\lambda^2+2\lambda-2}, \end{aligned}$$

$$r_{2h+1} = T^{2h+1}(r_0) = T^{2h+1}(-r_0) = \begin{bmatrix} -2 & -\lambda \\ \lambda & 2\lambda-2 \end{bmatrix}(-\alpha\lambda) = \frac{(2\alpha-1)\lambda}{-\alpha\lambda^2+2\lambda-2}.$$

We find that for $\alpha \in \left(\alpha_0, \frac{1}{2}\right)$ one has $-\alpha\lambda^2+2\lambda-2<0$ and $(\alpha-1)\lambda^2+2\lambda-2<0$. Thus,

$$\left|\frac{1}{r_{2h+1}}\right| - \left|\frac{1}{l_{2h+1}}\right| = \frac{-\alpha\lambda^2+2\lambda-2}{(2\alpha-1)\lambda} - \frac{(\alpha-1)\lambda^2+2\lambda-2}{(2\alpha-1)\lambda} = \frac{-2\alpha\lambda^2+\lambda^2}{(2\alpha-1)\lambda} = -\lambda\,.$$

Arguments completely analogous to the even case now give that that $l_{2h+2} = r_{2h+2}$ and in fact that $\mathcal{T}^{2h+2}(A_0) = \mathcal{T}^{2h+2}(D_0)$. $\square$

As stated in the introduction, [**11**] shows that the domain Ω_α is connected for all q and all $\alpha \in (\,1/2, 1/\lambda\,]$. Arguing as in Lemma IV.27 shows that Ω_{α_0} is not connected, thus part (i) of Theorem IV.3 follows for odd index q. From Lemma IV.32, Proposition IV.12 and Corollary IV.11 it follows that for $\alpha \in \left(\alpha_0, \frac{1}{2}\right]$ and odd q the

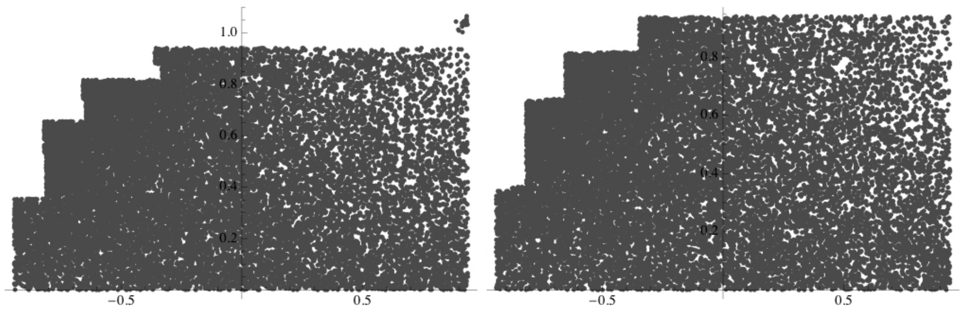


FIGURE 8. Change of topology at $\alpha = \alpha_0$: Simulations of the natural extension for $q = 9$; on the left $\alpha = \alpha_0 - 0.001$, on the right $\alpha = \alpha_0 + 0.001$.

entropy of the α-Rosen map equals that of the standard Rosen map. In Section 6 we finish the proof of part (ii) of Theorem IV.3.

Remark IV.33. As we displayed in each of Sections 3, 4 and 5, the key to successful quilting is equality of orbits of r_0 and of l_0 after a fixed number of steps. Compare this with the discussion in [**44**] relating eventual equality of these orbits and behavior of the entropy in the classical case. $\diamond$

6. Large α, by way of Dajani *et al.*

We finish the proof of part (ii) of Theorem IV.3 by appropriately interpreting results of [**11**] on $\alpha > \frac{1}{2}$.

6.1. Successful quilting for $\alpha \in (1/2, \omega_0]$. For even q and $\alpha \in (1/2, 1/\lambda]$, Theorem 2.2 of [**11**] shows that $r_p = l_p$, $d(r_{p-1}) = d(l_{p-1}) + 1$ and $\varepsilon(r_{p-1}) = -\varepsilon(l_{p-1})$. From this, just as above, one can in fact show that $\mathcal{T}^p(A_0) = \mathcal{T}^p(D_0)$ in these cases as well.

For odd q and $\alpha \in (1/2, \omega_0]$, Theorem 2.9 of [**11**] shows that $r_{2h+2} = l_{2h+2}$ and that just as in the even case their orbit predecessors have opposite signs and digits differing by 1. Thus, here one can show that $\mathcal{T}^{2h+2}(A_0) = \mathcal{T}^{2h+2}(D_0)$.

We find that for $\alpha \in (\alpha_0, \omega_0]$ the entropy of the α-Rosen map equals that of the standard Rosen map. To finish the prooaleorem IV.3 it remains to show that ω_0 cannot be replaced by a larger number.

As we must take $\alpha \leq \frac{1}{\lambda}$ to ensure that all the digits are positive, the optimality of ω_0 in the even case follows immediately.

6.2. Nearly successful quilting and unequal entropy. We now show that the entropy of T_α is *not* equal to that of $T_{1/2}$ for $\alpha_0 > \omega_0$ in the case of odd q. Indeed, for these values, the results of [**11**] show that although the natural extensions remain connected, the conditions for successful quilting are not fully satisfied.

Lemma IV.34. *With notation as above, suppose that there are distinct natural numbers k, k' such that*

$$\mathcal{T}_\alpha^k(A_0) = \mathcal{T}_\alpha^{k'}(D_0)\,.$$

Then the entropy of $(\mathcal{T}_\alpha, \Omega_\alpha, \overline{\mathcal{B}}_\alpha, \mu)$ differs from that of $(\mathcal{T}_{1/2}, \Omega_{1/2}, \overline{\mathcal{B}}_{1/2}, \mu)$.

Proof. If $k < k'$, then we can produce a new system $(\Omega'_\alpha, T'_\alpha, \overline{\mathcal{B}}'_\alpha, \mu)$ by inducing past $k' - k$ "copies" of A_0. This system can be shown to be isomorphic to $(\mathcal{T}_{1/2}, \Omega_{1/2}, \overline{\mathcal{B}}_{1/2}, \mu)$. But, by the Abramov formula [**1**], the induced system has entropy differing from that of the full system by a multiplicative factor equal to $k' - k$ times the measure of A_0. Thus, the entropy of $(\mathcal{T}_\alpha, \Omega_\alpha, \overline{\mathcal{B}}_\alpha, \mu)$ is less than that of the system of index $1/2$.

Similarly, if $k > k'$, we form a new system by inducing past the appropriate steps in the orbit of D_0 in the index $1/2$ system. This allows us to conclude that the entropy of $(\mathcal{T}_\alpha, \Omega_\alpha, \overline{\mathcal{B}}_\alpha, \mu)$ is greater than that of the system of index $1/2$ by a multiplicative factor of $k - k'$ times the measure of A_0. $\square$

The following is part of Theorem 2.9 of [**11**].

Lemma IV.35. (Dajani *et al.* [**11**]) *For odd q and $\alpha \in (\omega_0, 1/\lambda]$ there are distinct natural numbers k, k' such that $l_k = r_{k'}$ and that the T_α-digits of l_{k-1} and $r_{k'-1}$ differ by one.*

Corollary IV.36. *For odd q and $\alpha \in (\omega_0, 1/\lambda]$, the maps T_α and $T_{1/2}$ have distinct entropy values.*

Proof. From Lemma IV.35, one can show that $\mathcal{T}_\alpha^k(A_0) = \mathcal{T}_\alpha^{k'}(D_0)$. Lemma IV.34 then applies. $\square$

In fact, Theorem 2.9 of [**11**] shows that $k' = k + 1$; thus, from Lemmas IV.34 and IV.35, we see that the entropy of T_α for $\alpha \in (\omega_0, 1/\lambda]$ is decreasing, confirming the results for the case $q = 3$ of [**41**], see also [**37**].

If follows from Corollary IV.36 that ω_0 in part (ii) of Theorem IV.3 cannot be replaced by a larger number, which finishes the proof of this theorem.

V
An iterated LLL-algorithm for finding approximations with bounded Dirichlet coefficients

In this final chapter we study multi-dimensional continued fractions. We give an algorithm that finds a sequence of approximations with Dirichlet coefficients bounded by a constant only depending on the dimension. The algorithm uses the LLL-algorithm for lattice basis reduction. We present a version of the algorithm that runs in polynomial time of the input.

1. Introduction

We repeat Theorem I.50 and Definition I.52 from the Introduction.

Theorem V.1. *Let an $n \times m$ matrix A with entries $a_{ij} \in \mathbb{R} \backslash \mathbb{Q}$ be given and suppose that $1, a_{i1}, \ldots, a_{im}$ are linearly independent over $\mathbb{Q}$ for some i with $1 \leq i \leq n$. There exist infinitely many coprime m-tuples of integers $q_1, \ldots, q_m$ such that with $q = \max_j |q_j| \geq 1$, we have*

$$\max_i \|q_1\, a_{i1} + \cdots + q_m\, a_{im}\| < q^{\frac{-m}{n}}. \tag{V.2}$$

The exponent $\frac{-m}{n}$ of q is minimal.

Definition V.3. Let an $n \times m$ matrix A with entries $a_{ij} \in \mathbb{R} \setminus \mathbb{Q}$ be given. The *Dirichlet coefficient* of an m-tuple $q_1, \ldots, q_m$ is defined as

$$q^{\frac{m}{n}} \max_i \|q_1\, a_{i1} + \cdots + q_m\, a_{im}\|.$$

$\diamond$

Remark V.4. If $m = n = 1$, then the Dirichlet coefficient of an approximation $\frac{p}{q}$ for a is exactly the approximation coefficient $\Theta = q^2 \left| a - \frac{p}{q} \right|$, see Definition I.9 $\diamond$

The proof of the theorem does not give an efficient way of finding a series of approximations with a Dirichlet coefficient of 1. As mentioned in Section 1.5 there are many multi-dimensional algorithms that generalize one or more of the properties of the continued fraction algorithm, but there is no efficient algorithm known that guarantees to find a series of approximations with a Dirichlet coefficient of 1.

In 1982 the LLL-algorithm for lattice basis reduction was published in [**36**]. The authors noted that their algorithm could be used for finding Diophantine approximations of given rationals with Dirichlet coefficient only depending on the dimension; see (V.13).

Just [**24**] developed an algorithm based on lattice reduction that detects $\mathbb{Z}$-linear dependence in the a_i; in this case $m = 1$. If no such dependence is found her algorithm returns integers q with

$$\max_i \|qa_i\| \le c \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} q^{-1/(2n(n-1))},$$

where c is a constant depending on n. The exponent $-1/(2n(n-1))$ is larger than the Dirichlet exponent $-1/n$.

Lagarias [**32**] used the LLL-algorithm in a series of lattices to find good approximations for the case $m = 1$. Let $a_1, \ldots, a_n \in \mathbb{Q}$ and let there be a $Q \in \mathbb{N}$ with $1 \le Q \le N$ such that $\max_j ||Q\, a_j|| < \varepsilon$. Then Lagarias' algorithm on input $a_1, \ldots, a_n$ and N finds in polynomial time a q with $1 \le q \le 2^{\frac{n}{2}} N$ such that $\max_j ||q\, a_j|| \le \sqrt{5n} 2^{\frac{d-1}{2}} \varepsilon$. The main difference with our work is that Lagarias focuses on the quality $||q\, a_j||$, while we focus on Dirichlet coefficient $q^{\frac{1}{n}} ||q\, a_j||$. Besides that we also consider the case $m > 1$.

The main result of this chapter is an algorithm that by iterating the LLL-algorithm gives a series of approximations of given rationals with optimal Dirichlet exponent. Where the LLL-algorithm gives one approximation our dynamic algorithm gives a series of successive approximations. To be more precise: For a given $n \times m$-matrix A with entries $a_{ij} \in \mathbb{Q}$ and a given upper bound $q_{\max}$ the algorithm returns a sequence of m-tuples $q_1, \ldots, q_m$ such that for every Q with $2^{\frac{(m+n+3)(m+n)}{4m}} \le Q \le q_{\max}$ one of these m-tuples satisfies

$$\begin{aligned} &\max_j |q_j| \le Q \text{ and} \\ &\max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \le 2^{\frac{(m+n+3)(m+n)}{4n}} Q^{\frac{-m}{n}}. \end{aligned}$$

The exponent $-m/n$ of Q can not be improved and therefore we say that these approximations have optimal Dirichlet exponent.

Our algorithm is a multi-dimensional continued fraction algorithm in the sense that we work in a lattice basis and that we only interchange basis vectors and add integer multiples of basis vectors to another. Our algorithm differs from other multi-dimensional continued fraction algorithms in that the lattice is not fixed across the iterations.

In Lemma V.24 we show that if there exists an extremely good approximation, our algorithm finds a very good one. We derive in Theorem V.33 how the output of our algorithm gives a lower bound on the quality of possible approximations with coefficients up to a certain limit. If the lower bound is positive this proves that there do not exist linear dependencies with all coefficients q_i below the limit.

In Section 4 we show that a slightly modified version of our algorithm runs in polynomial time. In Section 5 we present some numerical data.

2. Systems of linear relations

Recall from Definition I.54 that a basis $b_1, \dots, b_r$ for a lattice L is *reduced* if

$$|\mu_{ij}| \le \frac{1}{2} \quad \text{for } 1 \le j < i \le r$$

and

$$|b_i^* + \mu_{ii-1} b_{i-1}^*|^2 \le \frac{3}{4} |b_{i-1}^*|^2 \quad \text{for } 1 \le i \le r,$$

where $|x|$ denotes the Euclidean length of the vector x.

We also repeat the following results from [**36**] already mentioned in the Introduction.

Proposition V.5. *Let $b_1, \dots, b_r$ be a reduced basis for a lattice L in $\mathbb{R}^r$. Then we have*

$$\begin{aligned} &(i) && |b_1| \le 2^{(r-1)/4} \left(\det(L)\right)^{1/r}, \\ &(ii) && |b_1|^2 \le 2^{r-1} |x|^2, \quad \textit{for every } x \in L, x \neq 0, \\ &(iii) && \prod_{i=1}^{r} |b_i| \le 2^{r(r-1)/4} \det(L). \end{aligned}$$

Proposition V.6. *Let $L \subset \mathbb{Z}^r$ be a lattice with a basis $b_1, b_2, \dots, b_r$, and let $F \in \mathbb{R}$, $F \ge 2$, be such that $|b_i|^2 \le F$ for $1 \le i \le r$. Then the number of arithmetic operations needed by the LLL-algorithm is $O(r^4 \log F)$ and the integers on which these operations are performed each have binary length $O(r \log F)$.*

We prove Lemma I.57 announced in the introduction.

Lemma V.7. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$ and an $\varepsilon \in (0,1)$ be given. Applying the LLL-algorithm to the basis formed by the columns of the $(m+n) \times (m+n)$-matrix*

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 & a_{11} & \dots & a_{1m} \\ 0 & 1 & \ddots & 0 & a_{21} & \dots & a_{2m} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & a_{n1} & \dots & a_{nm} \\ 0 & \dots & 0 & 0 & c & & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \\ 0 & \dots & 0 & 0 & 0 & & c \end{bmatrix}, \tag{V.8}$$

with $c = \left(2^{-\frac{m+n-1}{4}} \varepsilon\right)^{\frac{m+n}{m}}$ yields an m-tuple $q_1, \dots, q_m \in \mathbb{Q}$ with

$$\max_j |q_j| \le 2^{\frac{(m+n-1)(m+n)}{4m}} \varepsilon^{\frac{-n}{m}} \quad \textit{and} \tag{V.9}$$

$$\max_i \|q_1 a_{i1} + \dots + q_m a_{im}\| \le \varepsilon. \tag{V.10}$$

PROOF. The LLL-algorithm finds a reduced basis $b_1, \ldots, b_{m+n}$ for this lattice. Each vector in this basis can be written as

$$\begin{bmatrix} q_1 a_{11} + \cdots + q_m a_{1m} - p_1 \\ \vdots \\ q_1 a_{n1} + \cdots + q_m a_{nm} - p_n \\ cq_1 \\ \vdots \\ cq_m \end{bmatrix},$$

with $p_i \in \mathbb{Z}$ for $1 \le i \le n$ and $q_j \in \mathbb{Z}$ for $1 \le j \le m$.

Proposition V.5(i) gives an upper bound for the length of the first basis vector,

$$|b_1| \le 2^{\frac{m+n-1}{4}} c^{\frac{m}{m+n}}.$$

From this vector b_1 we find integers $q_1, \ldots, q_m$, such that

$$\text{(V.11)} \qquad \max_j |q_j| \quad \le \quad 2^{\frac{m+n-1}{4}} c^{\frac{-n}{m+n}} \text{ and}$$

$$\text{(V.12)} \qquad \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \quad \le \quad 2^{\frac{m+n-1}{4}} c^{\frac{m}{m+n}}.$$

Substituting $c = \left(2^{-\frac{m+n-1}{4}} \varepsilon\right)^{\frac{m+n}{m}}$ gives the results. $\square$

From equations (V.11) and (V.12) it easily follows that the m-tuple $q_1, \ldots, q_m$ satisfies

$$\text{(V.13)} \qquad \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \le 2^{\frac{(m+n-1)(m+n)}{4n}} q^{\frac{-m}{n}},$$

where $q = \max_j |q_j|$, so the approximation has a Dirichlet coefficient of at most $2^{\frac{(m+n-1)(m+n)}{4n}}$.

3. The Iterated LLL-algorithm

We iterate the LLL-algorithm over a series of lattices to find a sequence of approximations. We start with a lattice determined by a basis of the form (V.8). After the LLL-algorithm finds a reduced basis for this lattice, we decrease the constant c by dividing the last m rows of the matrix by a constant greater than 1. By doing so, ε is divided by this constant to the power $\frac{m}{m+n}$. We repeat this process until the upper bound (V.9) for q guaranteed by the LLL-algorithm exceeds a given upper bound $q_{\max}$. Motivated by the independence on ε of (V.13) we ease notation by fixing $\varepsilon = \frac{1}{2}$.

Define

$$\text{(V.14)} \qquad k' := \left\lceil -\frac{(m+n-1)(m+n)}{4n} + \frac{m \log_2 q_{\max}}{n} \right\rceil.$$

Iterated LLL-algorithm (ILLL)

Input
An $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$.
An upper bound $q_{\max} > 1$.

Output
For each integer k with $1 \le k \le k'$, see (V.15), we obtain a vector $q(k) \in \mathbb{Z}^m$ with

$$\max_j |q_j(k)| \quad \le \quad 2^{\frac{(m+n-1)(m+n)}{4m}}\, 2^{\frac{kn}{m}}, \tag{V.15}$$

$$\max_i \|q_1(k)\, a_{i1} + \cdots + q_m(k)\, a_{im}\| \quad \le \quad \frac{1}{2^k}. \tag{V.16}$$

Description of the algorithm

(1) Construct the basis matrix B as given in (V.8) from A.
(2) Apply the LLL-algorithm to B.
(3) Deduce $q_1, \ldots, q_m$ from the first vector in the reduced basis returned by the LLL-algorithm.
(4) Divide the last m rows of B by $2^{\frac{m+n}{m}}$
(5) Stop if the upper bound for q guaranteed by the algorithm (V.15) is larger than $q_{\max}$; else go to step 2.

Remark V.17. The number $2^{\frac{m+n}{m}}$ in step 4 may be replaced by $d^{\frac{m+n}{m}}$ for any real number $d > 1$. When we additionally set $\varepsilon = \frac{1}{d}$ this yields that

$$\max_j |q_j(k)| \quad \le \quad 2^{\frac{(m+n-1)(m+n)}{4m}}\, d^{\frac{kn}{m}} \qquad \text{and} \tag{V.18}$$

$$\max_i \|q_1(k)a_{i1} + \cdots + q_m(k)a_{im}\| \quad < \quad d^{-k}. \tag{V.19}$$

In the theoretical part of this chapter we always take $d = 2$ corresponding to our choice $\varepsilon = \frac{1}{2}$. $\diamond$

Lemma V.20. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$ and an upper bound $q_{\max} > 1$ be given. The number of times the ILLL-algorithm applies the LLL-algorithm on this input equals k' from* (V.14).

Proof. One easily derives the number of times we iterate by solving k from the stopping criterion (V.15)

$$q_{\max} \le 2^{\frac{(m+n-1)(m+n)}{4m}}\, 2^{\frac{kn}{m}},$$

$\square$

We define

$$c(k) = c(k-1)/2^{\frac{m+n}{m}} \text{ for } k > 1, \quad \text{where } c(1) = c \text{ as given in Lemma V.7.}$$

In iteration k we are working in the lattice defined by the basis in (V.8) with c replaced by $c(k)$.

Lemma V.21. *The k-th output, $q(k)$, of the ILLL-algorithm satisfies* (V.15) *and* (V.16).

Proof. In step k we use $c(k) = \left(2^{-\frac{m+n+3}{4}-k+1}\right)^{\frac{m+n}{m}}$. Substituting $c(k)$ for c in equations (V.11) and (V.12) yields (V.15) and (V.16), respectively. $\square$

The following theorem gives the main result mentioned in the introduction. The algorithm returns a sequence of approximations with all coefficients smaller than Q, optimal Dirichlet exponent and Dirichlet coefficient only depending on the dimensions m and n.

Theorem V.22. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$, and $q_{\max} > 1$ be given. The ILLL-algorithm finds a sequence of m-tuples $q_1, \ldots, q_m$ such that for every Q with $2^{\frac{(m+n+3)(m+n)}{4m}} \le Q \le q_{\max}$ one of these m-tuples satisfies*

$$\max_j |q_j| \le Q \text{ and}$$

$$\max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \le 2^{\frac{(m+n+3)(m+n)}{4n}} Q^{\frac{-m}{n}}.$$

Proof. Take $k \in \mathbb{N}$ such that

$$2^{\frac{(m+n+3)(m+n)}{4m}} \cdot 2^{\frac{(k-1)n}{m}} \le Q < 2^{\frac{(m+n+3)(m+n)}{4m}} \cdot 2^{\frac{kn}{m}}. \tag{V.23}$$

From Lemma V.21 we know that $q(k)$ satisfies the inequality

$$\max_j |q_j(k)| \le 2^{\frac{(m+n+3)(m+n)}{4m}} 2^{\frac{(k-1)n}{m}} \le Q.$$

From the right hand side of inequality (V.23) it follows that

$$\frac{1}{2^k} < 2^{\frac{(m+n+3)(m+n)}{4n}} Q^{\frac{-m}{n}}.$$

From Lemma V.21 and this inequality we derive that

$$\max_i \|q_1(k)\, a_{i1} + \cdots + q_m(k)\, a_{im}\| \le \frac{1}{2^k} < 2^{\frac{(m+n+3)(m+n)}{4n}} Q^{\frac{-m}{n}}.$$

$\square$

Proposition V.5(ii) guarantees that if there exists an extremely short vector in the lattice, then the LLL-algorithm finds a rather short lattice vector. We extend this result to the realm of successive approximations. In the next lemma we show that for every very good approximation, the ILLL-algorithm finds a rather good one not too far away from it.

Lemma V.24. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$, a real number $0 < \delta < 1$ and an integer $s > 1$ be given. If there exists an m-tuple $s_1, \ldots, s_m$ with*

$$s = \max_j |s_j| > 2^{\frac{(m+n-1)n}{4m}} \left(\frac{n\delta^2}{m}\right)^{\frac{n}{2(m+n)}} \tag{V.25}$$

and

$$\max_i \|s_1 a_{i1} + \cdots + s_m a_{im}\| \le \delta s^{\frac{-m}{n}}, \tag{V.26}$$

then applying the ILLL-algorithm with

$$q_{\max} \geq 2^{\frac{m^2+m(n-1)+4n}{4m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s \tag{V.27}$$

yields an m-tuple $q_1, \ldots, q_m$ with

$$\max_j |q_j| \leq 2^{\frac{m^2+m(n-1)+4n}{4m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s \tag{V.28}$$

and

$$\max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \leq 2^{\frac{m+n}{2}} \sqrt{n}\delta s^{\frac{-m}{n}}. \tag{V.29}$$

Proof. Let $1 \leq k \leq k'$ be an integer. Proposition V.5(ii) gives that for each $q(k)$ found by the algorithm

$$\begin{aligned} &\sum_{i=1}^{n} \|q_1(k)a_{i1} + \cdots + q_m(k)a_{im}\|^2 + c(k)^2 \sum_{j=1}^{m} q_j(k)^2 \\ &\leq 2^{m+n-1} \left(\sum_{i=1}^{n} \|s_1 a_{11} + \cdots + s_m a_{im}\|^2 + c(k)^2 \sum_{j=1}^{m} s_j^2 \right). \end{aligned}$$

From this and (V.25) and (V.26) it follows that

$$\max_i \|q_1(k)a_{i1} + \cdots + q_m(k)a_{im}\|^2 \leq 2^{m+n-1} \left(n\delta^2 s^{\frac{-2m}{n}} + c(k)^2 m s^2\right). \tag{V.30}$$

Take the smallest positive integer K such that

$$c(K) \leq \sqrt{\frac{n}{m}} \delta s^{-\frac{m+n}{n}}. \tag{V.31}$$

We find for step K from (V.30) and (V.31)

$$\max_i \|q_1(K)a_{i1} + \cdots + q_m(K)a_{im}\| \leq 2^{\frac{m+n}{2}} \sqrt{n}\delta s^{\frac{-m}{n}},$$

which gives (V.29).

We show that under assumption (V.27) the ILLL-algorithm makes at least K steps. We may assume $K > 1$, since the ILLL-algorithm always makes at least 1 step. From Lemma V.20 we find that if $q_{\max}$ satisfies

$$q_{\max} > 2^{\frac{Kn}{m}} 2^{\frac{(m+n-1)(m+n)}{4m}},$$

then the ILLL-algorithm makes at least K steps. Our choice of K implies

$$c(K-1) = \frac{c(1)}{2^{\frac{(m+n)(K-2)}{m}}} = \frac{2^{-\frac{(m+n+3)(m+n)}{4m}}}{2^{\frac{(m+n)(K-2)}{m}}} > \sqrt{\frac{n}{m}} \delta s^{-\frac{m+n}{n}},$$

and we obtain

$$2^{\frac{Kn}{m}} < 2^{-\frac{(m+n-5)n}{4m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s.$$

From this we find that

$$q_{\max} > 2^{\frac{m^2+m(n-1)+4n}{4m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s$$

is a satisfying condition to guarantee that the algorithm makes at least K steps.

Furthermore, either $2^{\frac{-(m+n)}{m}}\sqrt{\frac{n}{m}}\delta s^{-\frac{m+n}{n}} < c(K)$ or $K = 1$. In the former case we find from (V.11) that

$$\max_j |q_j(K)| \leq 2^{\frac{m+n-1}{4}} c(K)^{\frac{-n}{m+n}} < 2^{\frac{m+n-1}{4}} 2^{\frac{n}{m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s.$$

In the latter case we obtain from (V.11)

$$\max_j |q_j(1)| \leq 2^{\frac{m+n-1}{4}} c(1)^{\frac{-n}{m+n}} = 2^{\frac{m+n-1}{4}} 2^{\frac{(m+n+3)n}{4m}}$$

and, by (V.25),

$$2^{\frac{m+n-1}{4}} 2^{\frac{(m+n+3)n}{4m}} = 2^{\frac{m+n-1}{4}} 2^{\frac{n}{m}} 2^{\frac{(m+n-1)n}{4m}} < 2^{\frac{m+n-1}{4}} 2^{\frac{n}{m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s.$$

We conclude that for all $K \geq 1$

$$\max_j |q_j(K)| \leq 2^{\frac{m^2+m(n-1)+4n}{4m}} \left(\frac{m}{n\delta^2}\right)^{\frac{n}{2(m+n)}} s.$$

□

Note that from (V.28) and (V.29) it follows that

$$\text{(V.32)} \quad q^{\frac{m}{n}} \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \leq 2^{\frac{m^2+m(3n-1)+4n+2n^2}{4n}} m^{\frac{m}{2(m+n)}} (n\delta^2)^{\frac{n}{2(m+n)}},$$

where again $q = \max_j |q_j|$.

Theorem V.33. *Let an $n \times m$-matrix A with entries a_{ij} in $\mathbb{R}$ and $q_{\max} > 1$ be given. Assume that γ is such that for every m-tuple $q_1, \ldots, q_m$ returned by the ILLL-algorithm*

$$\text{(V.34)} \qquad q^{\frac{m}{n}} \max_i \|q_1 a_{i1} + \ldots q_m a_{im}\| > \gamma, \text{ where } q = \max_j |q_j|.$$

Then every m-tuple $s_1, \ldots, s_m$ with $s = \max_j |s_j|$ and

$$2^{\frac{(m+n-1)n}{4m}} \left(\frac{n\delta^2}{m}\right)^{\frac{n}{2(m+n)}} < s < 2^{-\frac{m^2+m(n-1)+4n}{4m}} \left(\frac{n\delta^2}{m}\right)^{\frac{n}{2(m+n)}} q_{\max}$$

satisfies

$$s^{\frac{m}{n}} \max_i \|s_1 a_{i1} + \cdots + s_m a_{im}\| > \delta,$$

with

$$\text{(V.35)} \qquad \delta = 2^{\frac{-(m+n)(m^2+m(3n-1)+4n+2n^2)}{4n^2}} m^{\frac{-m}{2n}} n^{\frac{-1}{2}} \gamma^{\frac{m+n}{n}}.$$

Proof. Assume that every vector returned by our algorithm satisfies (V.34) and that there exists an m-tuple $s_1, \ldots, s_m$ with $s = \max_j |s_j|$ such that

$$2^{\frac{(m+n-1)n}{4m}} \left(\frac{n\delta^2}{m}\right)^{\frac{n}{2(m+n)}} < s < 2^{-\frac{m^2+m(n-1)+4n}{4m}} \left(\frac{n\delta^2}{m}\right)^{\frac{n}{2(m+n)}} q_{\max}$$

$$\text{and} \quad s^{\frac{m}{n}} \max_i \|s_1 a_{i1} + \cdots + s_m a_{im}\| \leq \delta.$$

From the upper bound on s it follows that q_{max} satisfies (V.27). We apply Lemma V.24 and find that the algorithm finds an m-tuple $q_1, \ldots, q_m$ that satisfies (V.32). Substituting δ as given in (V.35) gives

$$q^{\frac{m}{n}} \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| \leq \gamma,$$

which is a contradiction with our assumption. □

4. A polynomial time version of the ILLL-algorithm

We have used real numbers in our theoretical results, but in a practical implementation of the algorithm we only use rational numbers. Without loss of generality we may assume that these numbers are in the interval $[0,1]$. In this section we describe the necessary changes to the algorithm and we show that this modified version of the algorithm runs in polynomial time.

As input for the rational algorithm we take

- the dimensions m and n,
- a rational number $\varepsilon \in (0,1)$,
- an integer M that is large compared to $\frac{(m+n)^2}{m} - \frac{m+n}{m} \log \varepsilon$,
- an $n \times m$-matrix A with entries $0 < a_{ij} \leq 1$, where each $a_{ij} = \frac{p_{ij}}{2^M}$ for some integer p_{ij},
- an integer $q_{\max} < 2^M$.

When we construct the matrix B in step 1 of the ILLL-algorithm we approximate c as given in (V.8) by a rational

$$\text{(V.36)} \qquad \hat{c} = \frac{\lceil 2^M c \rceil}{2^M} = \frac{\left\lceil 2^M \left(2^{-\frac{m+n-1}{4}} \varepsilon \right)^{\frac{m+n}{m}} \right\rceil}{2^M}.$$

Hence $c < \hat{c} \leq c + \frac{1}{2^M}$.

In iteration k we use a rational $\hat{c}(k)$ that for $k \geq 2$ is given by

$$\hat{c}(k) = \frac{\left\lceil 2^M \hat{c}(k-1) 2^{-\frac{m+n}{m}} \right\rceil}{2^M} \text{ and } \hat{c}(1) = \hat{c} \text{ as in } (V.36),$$

and we change step 4 of the ILLL-algorithm to 'multiply the last m rows of B by $\hat{c}(k-1)/\hat{c}(k)$'. The other steps of the rational iterated algorithm are as described in Section 3.

4.1. The running time of the rational algorithm.

Theorem V.37. *Let the input be given as described above. Then the number of arithmetic operations needed by the ILLL-algorithm and the binary length of the integers on which these operations are performed are both bounded by a polynomial in m, n and M.*

Proof. The number of times we apply the LLL-algorithm is not changed by rationalizing c, so we find the number of steps k' from Lemma V.20

$$k' = \left\lceil -\frac{(m+n-1)(m+n)}{4n} + \frac{m \log_2 q_{\max}}{n} \right\rceil < \left\lceil \frac{mM}{n} \right\rceil.$$

It is obvious that steps 1, 3, 4 and 5 of the algorithm are polynomial in the size of the input and we focus on the LLL-step. We determine an upper bound for the length of a basis vector used at the beginning of an iteration in the ILLL-algorithm.

In the first application of the LLL-algorithm the length of the initial basis vectors as given in (V.8) is bounded by

$$|b_i|^2 \le \max_j \left\{1, a_{1j}^2 + \cdots + a_{nj}^2 + m\hat{c}^2\right\} \le m+n, \quad \text{for } 1 \le i \le m+n\ .$$

where we use that $0 < a_{ij} < 1$ and $\hat{c} \le 1$.

The input of each following application of the LLL-algorithm is derived from the reduced basis found in the previous iteration by making some of the entries strictly smaller. Part (ii) of Proposition V.5 yields that for every vector b_i in a reduced basis it holds that

$$|b_i|^2 \le 2^{\frac{(m+n)(m+n-1)}{2}} (\det(L))^2 \prod_{j=1, j\neq i}^{m+n} |b_i|^{-2}.$$

The determinant of our starting lattice is given by $\hat{c}^m$ and the determinants of all subsequent lattices are strictly smaller. Every vector b_i in the lattice is at least as long as the shortest non-zero vector in the lattice. Thus for each i we have $|b_i|^2 \ge \frac{1}{2^M}$. Combining this yields

$$|b_i|^2 \le 2^{\frac{(m+n+2M)(m+n-1)}{2}} \hat{c}^{2m} \le 2^{\frac{(m+n+2M)(m+n-1)}{2}}$$

for every vector used as input for the LLL-step after the first iteration.

So we have

$$|b_i|^2 < \max\left\{m+n\,,\, 2^{\frac{(m+n+2M)(m+n-1)}{2}}\right\} = 2^{\frac{(m+n+2M)(m+n-1)}{2}} \tag{V.38}$$

for any basis vector that is used as input for an LLL-step in the ILLL-algorithm.

Proposition V.6 shows that for a given basis $b_1, \ldots, b_{m+n}$ for $\mathbb{Z}^{m+n}$ with $F \in \mathbb{R}$, $F \ge 2$ such that $|b_i|^2 \le F$ for $1 \le i \le m+n$ the number of arithmetic operations needed to find a reduced basis from this input is $O((m+n)^4 \log F)$. For matrices with entries in $\mathbb{Q}$ we need to clear denominators before applying this proposition. Thus for a basis with basis vectors $|b_i|^2 \le F$ and rational entries that can all be written as fractions with denominator 2^M the number of arithmetic operations is $O((m+n)^4 \log(2^{2M} F))$.

Combining this with (V.38) and the number of steps yields the proposition. □

4.2. Approximation results from the rational algorithm. Assume that the input matrix A (with entries $a_{ij} = \frac{p_{ij}}{2^M} \in \mathbb{Q}$) is an approximation of an $n \times m$-matrix $\mathcal{A}$ (with entries $\alpha_{ij} \in \mathbb{R}$), found by putting $a_{ij} = \frac{\lceil 2^M \alpha_{ij} \rceil}{2^M}$. In this subsection we derive the approximation results guaranteed by the rational iterated algorithm for the $\alpha_{ij} \in \mathbb{R}$.

According to (V.11) and (V.12) the LLL-algorithm applied with $\hat{c}$ instead of c guarantees to find an m-tuple $q_1, \ldots, q_m$ such that

$$\begin{aligned} q = \max_j |q_j| &\leq 2^{\frac{(m+n-1)(m+n)}{4m}} \varepsilon^{\frac{-n}{m}}, \\ &\text{and} \\ \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| &\leq 2^{\frac{m+n-1}{4}} \left(\left(2^{-\frac{m+n-1}{4}} \varepsilon \right)^{\frac{m+n}{m}} + \frac{1}{2^M} \right)^{\frac{m}{m+n}} \\ &\leq \varepsilon + 2^{\frac{(m+n-1)(m+n)-4Mm}{4(m+n)}}, \end{aligned}$$

the last inequality follows from the fact that $(x+y)^\alpha \leq x^\alpha + y^\alpha$ if $\alpha < 1$ and $x, y > 0$.

For the α_{ij} we find that

$$\begin{aligned} \max_i \|q_1 \alpha_{i1} + \cdots + q_m \alpha_{im}\| &\leq \max_i \|q_1 a_{i1} + \cdots + q_m a_{im}\| + mq2^{-M} \\ &\leq \varepsilon + 2^{\frac{m+n-1}{4} - \frac{Mm}{m+n}} + m\varepsilon^{\frac{-n}{m}} 2^{\frac{(m+n-1)(m+n)}{4m} - M}. \end{aligned}$$

On page 97 we have chosen M large enough to guarantee that the error introduced by rationalizing the entries is negligible.

We show that in every step the difference between $\hat{c}(k)$ and $c(k)$ is bounded by $\frac{2}{2^M}$.

Lemma V.39. *For each integer $k \geq 0$,*

$$c(k) \leq \hat{c}(k) < c(k) + \frac{1}{2^M} \sum_{i=0}^{k} 2^{-\frac{i(m+n)}{m}} < c(k) + \frac{2}{2^M}.$$

Proof. We use induction. For $k = 0$ we have $\hat{c}(0) = \frac{\lceil c(0) 2^M \rceil}{2^M}$ and trivially

$$c(0) \leq \hat{c}(0) < c(0) + \frac{1}{2^M}.$$

Assume that $c(k-1) \leq \hat{c}(k-1) < c(k-1) + \frac{1}{2^M} \sum_{i=0}^{k-1} 2^{-\frac{i(m+n)}{m}}$ and consider $\hat{c}(k)$. From the definition of $\hat{c}(k)$ and the induction assumption it follows that

$$\hat{c}(k) = \frac{\left\lceil \hat{c}(k-1)\, 2^{-\frac{m+n}{m}} 2^M \right\rceil}{2^M} \geq \frac{\hat{c}(k-1)}{2^{\frac{m+n}{m}}} \geq \frac{c(k-1)}{2^{\frac{m+n}{m}}} = c(k)$$

and

$$\begin{aligned} \hat{c}(k) = \frac{\left\lceil \hat{c}(k-1)\, 2^{-\frac{m+n}{m}} 2^M \right\rceil}{2^M} &< \frac{\hat{c}(k-1)}{2^{\frac{m+n}{m}}} + \frac{1}{2^M} \\ &< \frac{c(k-1) + \frac{1}{2^M} \sum_{i=0}^{k-1} 2^{-\frac{i(m+n)}{m}}}{2^{\frac{m+n}{m}}} + \frac{1}{2^M} \\ &= c(k) + \frac{1}{2^M} \sum_{i=0}^{k} 2^{-\frac{i(m+n)}{m}}. \end{aligned}$$

Finally note that $\sum_{i=0}^{k} 2^{-\frac{i(m+n)}{m}} < 2$ for all k. □

One can derive analogues of Theorem V.22, Lemma V.24 and Theorem V.33 for the polynomial version of the ILLL-algorithm by carefully adjusting for the introduced error. We do not give the details, since in practice this error is negligible.

5. Experimental data

In this section we present some experimental data from the rational ILLL-algorithm. In our experiments we choose the dimensions m and n and iteration speed d. We fill the $m \times n$ matrix A with random numbers in the interval $[0, 1]$ and repeat the entire ILLL-algorithm for a large number of these random matrices to find our results. First we look at the distribution of the approximation quality. Then we look at the growth of the denominators q found by the algorithm.

5.1. The distribution of the approximation qualities. For one-dimensional continued fractions the approximation coefficients Θ_k are defined as

$$\Theta_k = q_k^2 \left| a - \frac{p_k}{q_k} \right|,$$

where p_k/q_k is the nth convergent of a.

For the multi-dimensional case we define Θ_k in a similar way

$$\Theta_k = q(k)^{\frac{m}{n}} \max_i \|q_1(k)\, a_{i1} + \cdots + q_m(k)\, a_{im}\|. \tag{V.40}$$

5.1.1. *The one-dimensional case $m = n = 1$.* In [**5**] it was shown that for optimal continued fractions for almost all a one has that

$\lim_{N\to\infty} \frac{1}{N} \# \{1 \le n \le N : \Theta_n(x) \le z\} = F(z)$, where

$$F(z) = \begin{cases} \dfrac{z}{\log G}, & 0 \le z \le \frac{1}{\sqrt{5}}, \\[2ex] \dfrac{\sqrt{1-4z^2} + \log(G\frac{1-\sqrt{1-4z^2}}{2z})}{\log G}, & \frac{1}{\sqrt{5}} \le z \le \frac{1}{2}, \\[2ex] 1, & \frac{1}{2} \le z \le 1, \end{cases}$$

where $G = \frac{\sqrt{5}+1}{2}$.

As the name suggests, the optimal continued fraction algorithm gives the optimal approximation results. The denominators it finds, grow with maximal rate and all approximations with $\Theta < \frac{1}{2}$ are found.

We plot the distribution of the Θ's found by the ILLL-algorithm for $m = n = 1$ and $d = 2$ in Figure 1. The ILLL-algorithm might find the same approximation more than once. We see in Figure 1 that for $d = 2$ the distribution function differs depending on whether we leave in the duplicates or sort them out. With the duplicate approximations removed the distribution of Θ strongly resembles $F(z)$ of the optimal continued fraction. The duplicates that the ILLL-algorithm finds are usually good approximations: if they are much better than necessary they will also be an admissible solution in the next few iterations.

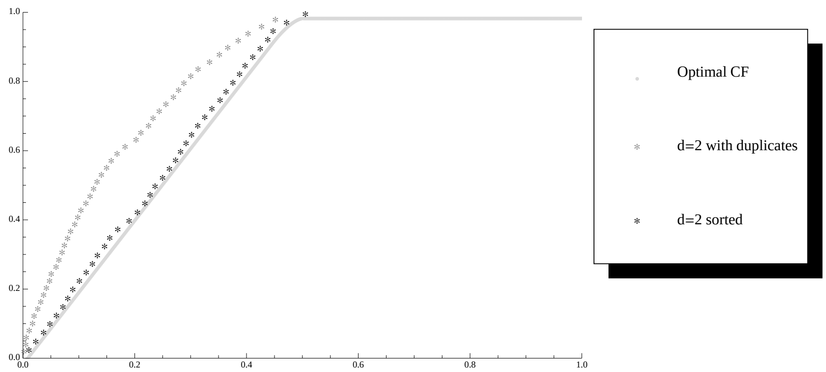


FIGURE 1. The distribution function for Θ from ILLL with $m = n = 1$ and $d = 2$, with and without the duplicate approximations, compared to the distribution function of Θ for optimal continued fractions.

For larger d we do not find so many duplicates, because the quality has to improve much more in every step; also see Figure 2 for an example with $d = 64$.

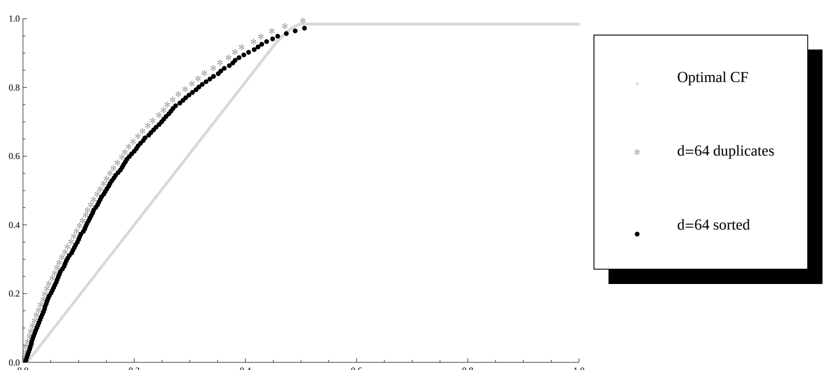


FIGURE 2. The distribution function for Θ from ILLL with $m = n = 1$ and $d = 64$, with and without the duplicate approximations, compared to the distribution function of Θ for optimal continued fractions.

From now on we remove duplicates from our results.

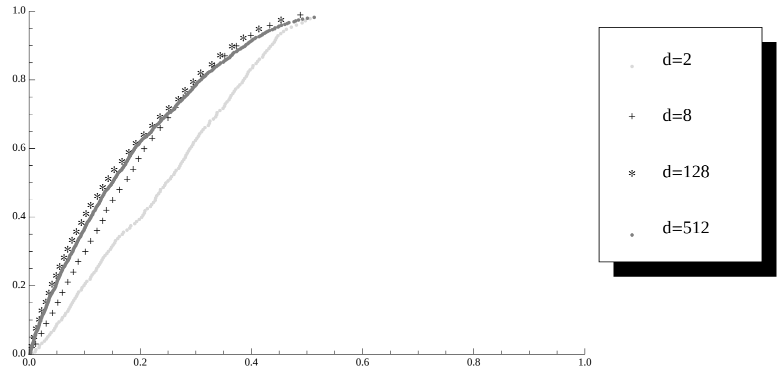


FIGURE 3. The distribution function for Θ from ILLL (with duplicates removed) with $m = n = 1$ and various values of d.

5.2. The multi-dimensional case. In this section we show some results for the distribution of the Θ's found by the ILLL-algorithm. For fixed m and n there also appears to be a limit distribution for Θ as d grows. See Figure 4 for an example with $m = 3$ and $n = 2$, and compare this with Figure 3. In this section we fix $d = 512$.

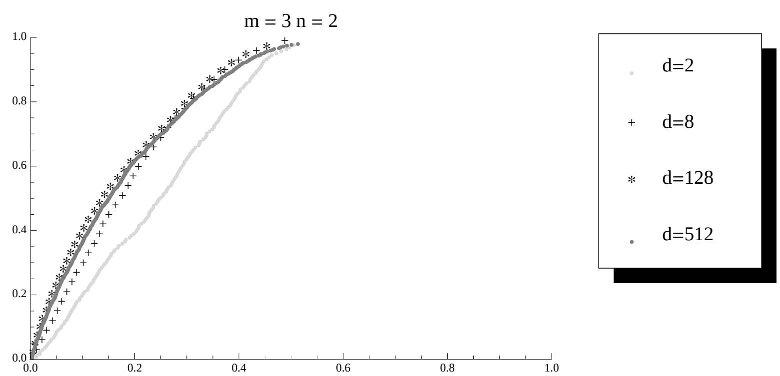


FIGURE 4. The distribution function for Θ from ILLL with $m = 3$ and $n = 2$ for $d = 2, 8, 128$ and 512.

In Figure 5 we show some distributions for cases where either m or n is 1.

In Figure 6 we show some distributions for cases where $m = n$.

Remark V.41. Very rarely the ILLL-algorithm returns an approximation with $\Theta > 1$, but this is not visible in the images in this section. $\diamond$

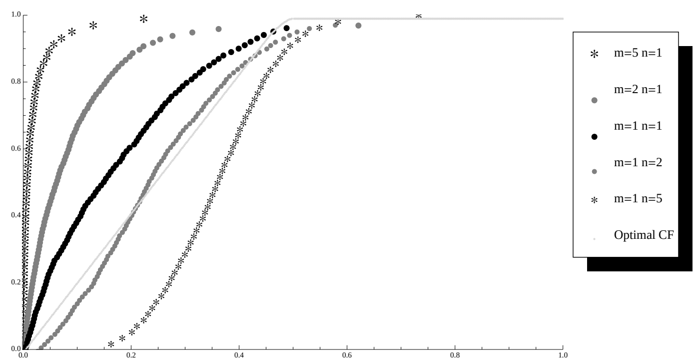


FIGURE 5. The distribution for Θ from ILLL when either $m = 1$ or $n = 1$.

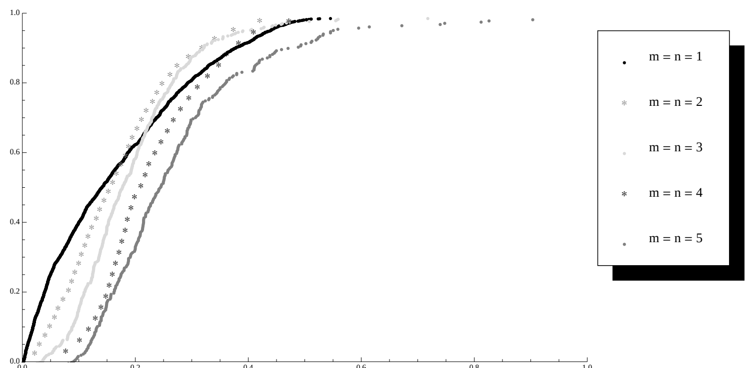


FIGURE 6. The distribution of Θ from ILLL when $m = n$.

5.3. The denominators q. For regular continued fractions, the denominators grow exponentially fast, to be more precise, for almost all x we have that

$$\lim_{k\to\infty} q_k^{1/k} = e^{\frac{\pi^2}{12\log 2}},$$

see Section 3.5 of [**10**].

For nearest integer continued fractions the constant $\frac{\pi^2}{12\log 2}$ is replaced by $\frac{\pi^2}{12\log G}$ with $G = \frac{\sqrt{5}+1}{2}$. For multi-dimensional continued fraction algorithms little is known about the distribution of the denominators q_j. Lagarias defined in [**31**] the notion of a best simultaneous Diophantine approximation and showed that for the ordered denominators $1 = q_1 < q_2 < \dots$ of best approximations for $a_1, \dots, a_n$ it holds that

$$\lim_{k\to\infty} \inf q_k^{1/k} \geq 1 + \frac{1}{2^{n+1}}.$$

We look at the growth of the denominators $q = \max_j |q_j|$ that are found by the ILLL-algorithm. Dirichlet's Theorem V.1 suggests that if q grows exponentially with a rate of m/n, then infinitely many approximation with Dirichlet coefficient smaller than 1 can be found. In the iterated LLL-algorithm it is guaranteed by (V.15) that $q(k)$ is smaller than a constant times $d^{\frac{kn}{m}}$. Our experiments indicate that $q(k)$ is about $d^{\frac{kn}{m}}$, or equivalently that $e^{\frac{m \log q_k}{k\, n}}$ is about d; see Figure 7 which gives a histogram of solutions that satisfy $e^{\frac{m \log q_k}{k\, n}} = x$.

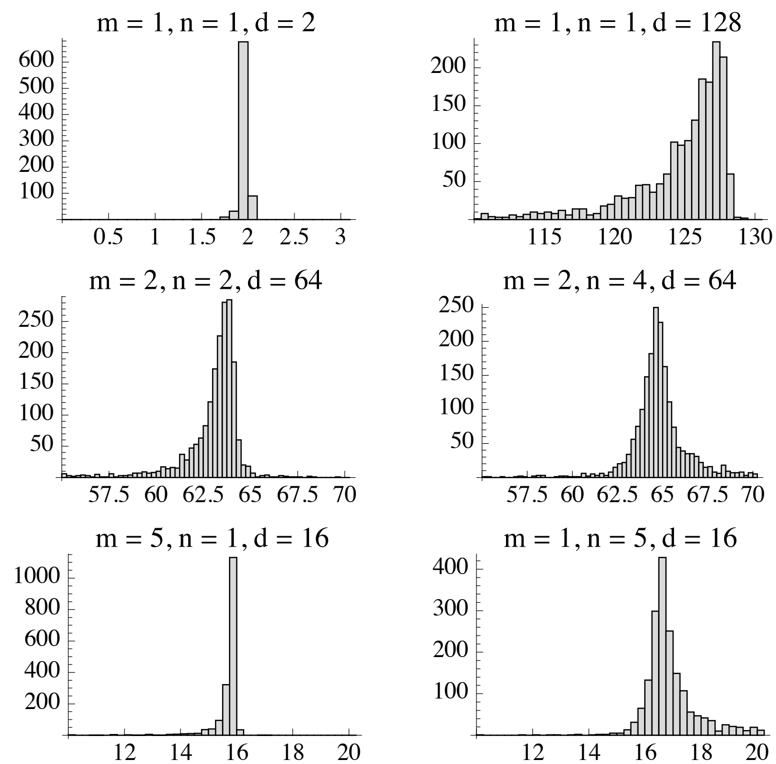


FIGURE 7. Histograms of $e^{\frac{m \log q(k)}{k\, n}}$ for various values of m, n and d. In these experiments we used $q_{\max} = 10^{40}$ and repeated the ILLL-algorithm $\left\lfloor \frac{2000}{k'} \right\rfloor$ times, with k' from Lemma V.20.

References

[1] L. M. Abramov. The entropy of a derived automorphism. *Dokl. Akad. Nauk SSSR*, 128:647–650, 1959. (Page 88).

[2] F. Bagemihl and J. R. McLaughlin. Generalization of some classical theorems concerning triples of consecutive convergents to simple continued fractions. *J. Reine Angew. Math.*, 221:146–149, 1966. (Page 17).

[3] É. Borel. Contribution à l'étude des fonctions méromorphes. *Ann. Sci. École Norm. Sup. (3)*, 18:211–239, 1901. (Page 3).

[4] W. Bosma, H. Jager, and F. Wiedijk. Some metrical observations on the approximation by continued fractions. *Indag. Math.*, 45(3):281–299, 1983. (Pages 5, 37, 67).

[5] W. Bosma and C. Kraaikamp. Metrical theory for optimal continued fractions. *J. Number Theory*, 34(3):251–270, 1990. (Page 100).

[6] A. J. Brentjes. *Multidimensional continued fraction algorithms*, volume 145 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam, 1981. (Page 13).

[7] V. Brun. En generalisation av kjedebroken i+ii. *Skr. Vid. Selsk. Kristiana, Mat. Nat. 6 (1919) and 6 (1920)*, 1919. (Page 13).

[8] R. M. Burton, C. Kraaikamp, and T. A. Schmidt. Natural extensions for the Rosen fractions. *Trans. Amer. Math. Soc.*, 352(3):1277–1298, 2000. (Pages 12, 53, 63, 71, 72, 72, 76, 81, 81, 81, 81, 82, 83, 85, 85, 85).

[9] T. W. Cusick and M. E. Flahive. *The Markoff and Lagrange spectra*, volume 30 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1989. (Page 35).

[10] K. Dajani and C. Kraaikamp. *Ergodic theory of numbers*, volume 29 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 2002. (Pages 1, 1, 4, 5, 5, 6, 39, 103).

[11] K. Dajani, C. Kraaikamp, and W. Steiner. Metrical theory for α-Rosen fractions. *J. Eur. Math. Soc. (JEMS)*, 11:1259–1283, 2009. (Pages 10, 11, 11, 38, 40, 40, 41, 41, 54, 56, 58, 58, 59, 61, 61, 63, 68, 68, 68, 71, 71, 74, 76, 85, 86, 87, 87, 87, 87, 88, 88, 88).

[12] W. Doeblin. Remarques sur la théorie métrique des fractions continues. *Compositio Math.*, 7:353–371, 1940. (Page 37).

[13] H. R. P. Ferguson and R. W. Forcade. Generalization of the Euclidean algorithm for real numbers to all dimensions higher than two. *Bull. Amer. Math. Soc. (N.S.)*, 1(6):912–914, 1979. (Page 13).

[14] L. R. Ford. *An introduction to the theory of automorphic functions*. Edinburgh mathematical tracts, no. 6. G. Bell, London, 1915, `http://www.archive.org/details/introductiontoth00forduoft`. (Pages 9, 50).

[15] M. Fujiwara. Bemerkung zur Theorie der Approximation der irrationalen Zahlen durch rationale Zahlen. *Tôhoku Math. J.*, 14:109–115, 1918. (Page 17).

[16] A. Haas and C. Series. The Hurwitz constant and Diophantine approximation on Hecke groups. *J. London Math. Soc. (2)*, 34(2):219–234, 1986. (Pages 11, 37, 37).

[17] Y. Hartono and C. Kraaikamp. Tong's spectrum for semi-regular continued fraction expansions. *Far East J. Math. Sci. (FJMS)*, 13(2):137–165, 2004. (Pages 36, 58, 59, 63).

[18] D. Hensley. *Continued fractions*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. (Pages 1, 11).

[19] A. Hurwitz. Über die angenäherte Darstellung der Zahlen durch rationale Brüche. *Math. Ann.*, 44(2-3):417–436, 1894. (Page 3).

[20] M. Iosifescu and C. Kraaikamp. *Metrical theory of continued fractions*, volume 547 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2002. (Page 5).

[21] S. Ito. On Legendre's theorem related to Diophantine approximations. In *Séminaire de Théorie des Nombres, 1987–1988 (Talence, 1987–1988)*, pages Exp. No. 44, 19. Univ. Bordeaux I, Talence, 1988. (Page 35).

[22] C. Jacobi. Allgemeine Theorie der kettenbruchähnlichen Algorithmen. *J. Reine Angew. Math.*, 69:29–64, 1868. (Page 13).

[23] H. Jager and C. Kraaikamp. On the approximation by continued fractions. *Indag. Math.*, 51(3):289–307, 1989. (Page 8).

[24] B. Just. Generalizing the continued fraction algorithm to arbitrary dimensions. *SIAM J. Comput.*, 21(5):909–926, 1992. (Pages 13, 90).

[25] M. Keane and M. Smorodinsky. Bernoulli schemes of the same entropy are finitarily isomorphic. *Ann. of Math. (2)*, 109(2):397–406, 1979. (Page 6).

[26] A. Y. Khinchin. *Continued fractions*. The University of Chicago Press, Chicago, Ill.-London, 1964. (Pages 1, 2).

[27] C. Kraaikamp. On the approximation by continued fractions. II. *Indag. Math. (N.S.)*, 1(1):63–75, 1990. (Page 17).

[28] C. Kraaikamp. On symmetric and asymmetric Diophantine approximation by continued fractions. *J. Number Theory*, 46(2):137–157, 1994. (Page 17).

[29] C. Kraaikamp, H. Nakada, and T. A. Schmidt. Metric and arithmetic properties of mediant-Rosen maps. *Acta Arith.*, 137(4):295–324, 2009. (Page 37).

[30] C. Kraaikamp, T. A. Schmidt, and I. Smeets. Tong's spectrum for Rosen continued fractions. *J. Théor. Nombres Bordeaux*, 19(3):641–661, 2007. (Pages 36, 40, 50, 55, 58, 66).

[31] J. C. Lagarias. Best simultaneous Diophantine approximations. I. Growth rates of best approximation denominators. *Trans. Amer. Math. Soc.*, 272(2):545–554, 1982. (Page 103).

[32] J. C. Lagarias. The computational complexity of simultaneous Diophantine approximation problems. *SIAM J. Comput.*, 14(1):196–209, 1985. (Page 90).

[33] J. C. Lagarias. Geodesic multidimensional continued fractions. *Proc. London Math. Soc. (3)*, 69(3):464–488, 1994. (Page 13).

[34] A. M. Legendre. Essai sur la théorie des nombres, 1798. (Page 2).

[35] G. Lejeune Dirichlet. *Mathematische Werke. Bände I, II.* Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften von L. Kronecker. Chelsea Publishing Co., Bronx, N.Y., 1969. (Pages 12, 17).

[36] A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann.*, 261(4):515–534, 1982. (Pages 13, 14, 15, 90, 91).

[37] L. Luzzi and S. Marmi. On the entropy of Japanese continued fractions. *Discrete Contin. Dyn. Syst.*, 20(3):673–711, 2008. (Pages 9, 71, 88).

[38] T. E. McKinney. Concerning a Certain Type of Continued Fractions Depending on a Variable Parameter. *Amer. J. Math.*, 29(3):213–278, 1907. (Page 9).

[39] P. Moussa, A. Cassa, and S. Marmi. Continued fractions and Bruno functions. *J. Comput. Appl. Math.*, 105(1-2):403–415, 1999. Continued fractions and geometric function theory (CONFUN) (Trondheim, 1997). (Pages 9, 71, 77).

[40] M. Müller. Über die Approximation reeller Zahlen durch die Näherungsbrüche ihres regelmäßigen Kettenbruches. *Arch. Math.*, 6:253–258, 1955. (Page 17).

[41] H. Nakada. Metrical theory for a class of continued fraction transformations and their natural extensions. *Tokyo J. Math.*, 4(2):399–426, 1981. (Pages 5, 9, 58, 71, 77, 78, 88).

[42] H. Nakada. On the Lenstra constant associated to the Rosen continued fractions. *To appear in J. Eur. Math. Soc. (JEMS). Available at* http://arxiv.org/abs/math/0705.3756, 2007. (Pages 12, 37, 37, 68, 69, 72).

[43] H. Nakada, S. Ito, and S. Tanaka. On the invariant measure for the transformations associated with some real continued-fractions. *Keio Engrg. Rep.*, 30(13):159–175, 1977. (Page 5).

[44] H. Nakada and R. Natsui. The non-monotonicity of the entropy of α-continued fraction transformations. *Nonlinearity*, 21(6):1207–1225, 2008. (Pages 9, 71, 78, 87).

[45] P. Q. Nguyen and B. E. Vallée. *The LLL Algorithm. Survey and Applications.* Springer, Berlin, Germany, 2009. (Page 15).

[46] D. Ornstein. Ornstein theory. *Scholarpedia*, 3(3):3957, 2008. (Page 6).

[47] O. Perron. Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus. *Math. Ann.*, 64(1):1–76, 1907. (Page 13).

[48] O. Perron. *Die Lehre von den Kettenbrüchen*. Chelsea Publishing Co., New York, N. Y., 1950. 2d ed. (Page 1).

[49] V. A. Rohlin. Exact endomorphisms of Lebesgue spaces. *Amer. Math. Soc. Transl. Series 2*, pages 1–36., 1964. (Pages 5, 6, 77, 77).

[50] D. Rosen. A class of continued fractions associated with certain properly discontinuous groups. *Duke Math. J.*, 21:549–563, 1954. (Pages 9, 10).

[51] W. M. Schmidt. *Diophantine approximation*, volume 785 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980. (Pages 1, 13).

[52] F. Schweiger. *Ergodic theory of fibred systems and metric number theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1995. (Page 76).

[53] F. Schweiger. *Multidimensional continued fractions*. Oxford Science Publications. Oxford University Press, Oxford, 2000. (Page 13).

[54] P. C. Shields. *The ergodic theory of discrete sample paths*, volume 13 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996. (Page 6).

[55] J. C. Tong. The conjugate property of the Borel theorem on Diophantine approximation. *Math. Z.*, 184(2):151–153, 1983. (Page 17).

[56] J. C. Tong. Segre's theorem on asymmetric Diophantine approximation. *J. Number Theory*, 28(1):116–118, 1988. (Page 17).

[57] J. C. Tong. Symmetric and asymmetric Diophantine approximation of continued fractions. *Bull. Soc. Math. France*, 117(1):59–67, 1989. (Pages 7, 17, 17).

[58] J. C. Tong. Diophantine approximation by continued fractions. *J. Austral. Math. Soc. Ser. A*, 51(2):324–330, 1991. (Pages 7, 17).

[59] J. C. Tong. Approximation by nearest integer continued fractions. *Math. Scand.*, 71(2):161–166, 1992. (Pages 8, 36).

[60] J. C. Tong. Approximation by nearest integer continued fractions. II. *Math. Scand.*, 74(1):17–18, 1994. (Pages 8, 36).

[61] J. C. Tong. Symmetric and asymmetric Diophantine approximation. *Chinese Ann. Math. Ser. B*, 25(1):139–142, 2004. (Pages 18, 32).

The page numbers at the end of an entry indicate the pages in this thesis where the reference occurs.

Samenvatting

Deze samenvatting is bedoeld voor mijn moeder – en alle andere lezers die niet veel van wiskunde weten, maar wel graag willen zien waar ik de afgelopen jaren aan heb gewerkt. Wiskundigen verwijs ik graag door naar Hoofdstuk I. Daar staan definities van de gebruikte begrippen, klassieke stellingen en de belangrijkste resultaten uit dit proefschrift.

1. Hoeveel decimalen van π ken je?

Han, o lief, o zoete hartedief...

Bovenstaande dichtregel is niet alleen een liefdesverklaring, het is ook een ezelsbruggetje om de eerste decimalen van $\pi = 3{,}14159265358\ldots$ (de verhouding tussen de omtrek van een cirkel en zijn diameter) te onthouden. Tel maar eens het aantal letters van de woorden. Er zijn veel meer van dit soort ezelsbruggetjes in allerlei talen:

How	I	wish	I	could	recollect	pi	easily	today	...			
Sol	y	Luna	y	Cielo	proclaman	al	Divino	Autor	del	Cosmo	...	
Wat	u	door	’n	goede	ezelsbrug	te	kennen	immer	met	gemak	onthoudt	...
How	I	want	a	drink,	alcoholic	of	course,	after	the	heavy	lectures	...
3	1	4	1	5	9	2	6	5	3	5	8 ...	

Eigenlijk heeft π oneindig veel decimalen achter de komma. Wat betekent het als je alleen de eerste vijf decimalen van π gebruikt? Je benadert π dan met $\frac{314159}{100000} = 3{,}14159$.

Misschien herinner je je een andere benadering van π die vaak gebruikt wordt op school: $\frac{22}{7} \approx 3{,}14285714$. Deze breuk met een heel kleine noemer (7) benadert de eerste twee decimalen van π. Archimedes gebruikte deze benadering al rond 200 voor Christus, maar het kan nog veel beter. Bijvoorbeeld met de breuk $\frac{355}{113}$. Die is ongeveer gelijk aan 3,14159292 en benadert π op maar liefst zes decimalen. Deze benadering is zo goed, dat geen enkele breuk met noemer kleiner dan 16604 dichter bij π ligt. Hulde dus voor de Chinese wiskundige Zu Chongzhi die in 480 (zo’n vier jaar na de val van het Romeinse rijk) met veel moeite deze benadering vond.

Archimedes en Chongzhi vonden hun benaderingen voor π door veelhoeken in cirkels te tekenen. Maar je kunt voor elk willekeurig getal goede benaderingen maken met *kettingbreuken*.

2. Wat is een kettingbreuk?

Een kettingbreuk is een breuk in een breuk in een breuk, enzovoorts. Zo ziet de kettingbreuk voor π er bijvoorbeeld uit:

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\dots}}}}.$$

In de breuk heb je steeds een 1, een deelstreep, een positief geheel getal en dan weer een nieuwe breuk die begint met een 1. Dit soort kettingbreuken noemen we *reguliere* kettingbreuken. We noteren het getal voor de breuk met a_0, voor π geldt dus $a_0 = 3$. De positieve gehele getallen in de breuk noteren we als $a_1, a_2, a_3, \ldots$. In het voorbeeld hierboven geldt $a_1 = 7, a_2 = 15, a_3 = 1$ en $a_4 = 292$.

Een getal dat geen breuk is, kun je op precies één manier schrijven als een oneindig lange kettingbreuk. Zulke getallen noemen we *irrationaal*.[1]

3. Hoe haal je benaderingen uit een kettingbreuk?

Als je de oneindig lange kettingbreuk afkapt, krijg je een *benaderingsbreuk*. Je neemt alleen het eerste deel en schrijft dat als een gewone breuk. Laten we eens kijken wat we dan vinden voor π. We noteren voortaan alleen de eerste acht cijfers achter de komma.

$$\begin{array}{llcll}
 & \pi & & & \approx \mathbf{3{,}14159265} \\
1) & 3+\frac{1}{7} & = & \frac{22}{7} & \approx \mathbf{3{,}14}285714 \\
2) & 3+\frac{1}{7+\frac{1}{15}} & = & \frac{333}{106} & \approx \mathbf{3{,}1415}0943 \\
3) & 3+\frac{1}{7+\frac{1}{15+\frac{1}{1}}} & = & \frac{355}{113} & \approx \mathbf{3{,}141592}92
\end{array}$$

We zien hier de twee eeuwenoude benaderingen $\frac{22}{7}$ en $\frac{355}{113}$ tevoorschijn komen.

Voor een willekeurig getal x schrijven we de benaderingen die we op deze manier vinden als $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$. In het algemeen noemen we de benadering die we vinden door de eerste n termen van de kettingbreuk te gebruiken $\frac{p_n}{q_n}$.

[1] Kijk eens voor een mooi bewijs dat $\sqrt{2}$ geen breuk is op `http://nl.wikipedia.org/wiki/Bewijs_dat_wortel_2_irrationaal_is`.

4. Hoe maak je zo'n kettingbreuk?

Neem een willekeurig irrationaal getal dat je wilt benaderen (zoals net bijvoorbeeld $\pi = 3{,}14152965\ldots$). Het deel voor de komma is al een geheel getal, dus dat hoef je niet te benaderen. Noem het niet-gehele deel achter de komma x. Het recept om die kettingbreuk te maken is eenvoudiger dan dat voor saltimbocca:

> Bereken $\frac{1}{x}$ en neem het gehele deel van $\frac{1}{x}$ als volgende getal a in je kettingbreuk. Zet $x = \frac{1}{x} - a$ en begin opnieuw.

Wiskundigen noemen zo'n recept een *algoritme*.

Als voorbeeld maken we de kettingbreuk voor $\pi = 3{,}14159265\ldots$. We passen het recept toe op $\pi - 3 = 0{,}14159265\ldots$.

Stap 1) geeft $\frac{1}{0{,}14159265\ldots} = 7{,}06251330\ldots$, dus $a_1 = 7$.
We zetten $x = 0{,}06251330\ldots$.

Stap 2) geeft $\frac{1}{0{,}06251330\ldots} = 15{,}99659440\ldots$, dus $a_2 = 15$.
We zetten $x = 0{,}99659440\ldots$.

Stap 3) geeft $\frac{1}{0{,}99659441\ldots} = 1{,}00341723\ldots$, dus $a_3 = 1$.
We zetten $x = 0{,}00341723\ldots$.

Enzovoorts.

We vinden $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \ldots}}}$.

Je kunt dit voor elk irrationaal getal doen en nu bijvoorbeeld zelf narekenen dat voor $\sqrt{2} \approx 1{,}41421356$ de kettingbreuk wordt gegeven door

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}.$$

5. Waarom werkt het recept om kettingbreuken te maken?

Wiskundigen gebruiken in plaats van het bovenstaande recept graag een functie $T(x)$. Je kunt het recept namelijk ook beschrijven als het steeds herhalen van $T(x) = \frac{1}{x} - a(x)$, oftewel $x = \frac{1}{a(x)+T(x)}$, waarbij $a(x)$ het gehele deel van $\frac{1}{x}$ is.

Voor de eerste stap geldt dus:

$$T(x) = \frac{1}{x} - a_1 \quad \text{oftewel} \quad x = \frac{1}{a_1 + T(x)}$$

In de volgende stap pas je T toe op $T(x)$ en krijg je

$$T(T(x)) = \frac{1}{T(x)} - a(T(x)) = \frac{1}{\frac{1}{x} - a_1} - a_2 \quad \text{oftewel} \quad x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T(T(x))}}.$$

Zo ga je verder en je ziet hoe de kettingbreuk zich na elke stap een stukje verder uitrolt.

6. Wat is een goede benadering?

In het voorbeeld hierboven kwam je met elke stap dichter bij π. Dat is altijd zo bij het kettingbreukalgoritme: elke benadering $\frac{p_n}{q_n}$ ligt dichter bij x dan de vorige benadering $\frac{p_{n-1}}{q_{n-1}}$. Of zoals wiskundigen het schrijven

$$\left|x - \frac{p_n}{q_n}\right| < \left|x - \frac{p_{n-1}}{q_{n-1}}\right|;$$

waarbij $|\,|$ staat voor het nemen van de absolute waarde. De limiet van de benaderingsbreuken $\frac{p_n}{q_n}$ is x.

Neem nu een willekeurige breuk $\frac{p}{q}$ die dicht bij x ligt. Wanneer noem je deze breuk een goede benadering van x? Het ligt voor de hand om de noemer van de breuk mee te nemen in de kwaliteit. Het is natuurlijk veel makkelijker om dicht bij x te komen als je een breuk met noemer 100.000.000 neemt, dan wanneer je een breuk met noemer 10 gebruikt.

Een veel gebruikte maat voor de kwaliteit van een benadering is de noemer in het kwadraat keer het verschil tussen x en de breuk $\frac{p}{q}$, in wiskundige notatie

$$q^2 \left|x - \frac{p}{q}\right|.$$

Een benadering is goed als de kwaliteit zo klein mogelijk is: dan heb je zowel een kleine noemer als een korte afstand tot x. In tegenstelling tot de Consumentenbond geven we dus breuken met een *lage* kwaliteit het predicaat 'beste keus'.

Het blijkt dat hoe groter een getal a in de kettingbreuk is, des te beter de benadering is die je krijgt door net daarvoor af te kappen. In het voorbeeld met π hebben we $a_2 = 15, a_3 = 1$ en $a_4 = 292$. Als we afkappen voor $a_2 = 15$ dan krijgen we als benadering $\frac{22}{7}$ met kwaliteit van ongeveer 0,062. Afkappen voor $a_3 = 1$ geeft $\frac{333}{106}$ en die doet het met een kwaliteit van 0,94 minder goed. Als we afkappen voor de grote $a_4 = 292$, dan krijgen we Chongzi's benadering $\frac{355}{113}$ en die heeft een spectaculaire kwaliteit van 0,0034.

7. Waar vind je die goede benaderingen?

We zagen hierboven dat de kwaliteit van de benaderingen steeds tussen de nul en één lag. Dit is altijd waar, voor elke kettingbreukbenadering van welk irrationaal getal dan ook. Het is helemaal niet zo vanzelfsprekend dat een benadering zulke goede kwaliteit heeft. De benadering $\frac{314159}{100000}$ voor π heeft bijvoorbeeld kwaliteit 26535,9.

Het is dus bijzonder dat het kettingbreukalgoritme alleen maar benaderingen met kwaliteit kleiner dan één vindt.

Kan het nu zo zijn dat er breuken bestaan die ontzettend goede benaderingen zijn, maar die het kettingbreukalgoritme per ongeluk níet vindt? Legendre bewees in 1798 (het jaar dat Napoleons troepen Egypte binnentrokken) dat dit niet kan.

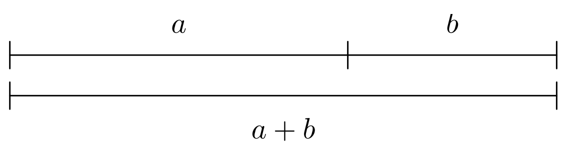
Stelling 1. *Als $\frac{p}{q}$ een breuk is die x benadert met een kwaliteit van minder dan $\frac{1}{2}$, dan wordt deze breuk gevonden door het kettingbreukalgoritme.*

Kortom: alle echt goede benaderingen worden gevonden door het kettingbreukalgoritme.

Borel bewees in 1905 (het jaar waarin Albert Einstein zijn speciale relativiteitstheorie publiceerde) dat bovendien één in elke drie opeenvolgende benaderingen heel goed is.

Stelling 2. *Voor elke irrationale x en voor elke drie willekeurige achtereenvolgende kettingbreukbenaderingen geldt dat het minimum van de drie bijbehorende kwaliteiten kleiner is dan* $\frac{1}{\sqrt{5}} \approx 0{,}44721$. *De constante kan niet vervangen worden door een kleinere.*

Voor elke constante kleiner dan $\frac{1}{\sqrt{5}}$ bestaan er getallen x waarvoor je geen drie opeenvolgende benaderingen kunt aanwijzen die elk kwaliteit kleiner dan die constante hebben. De *gulden snede* φ is een voorbeeld van zo'n getal waarbij het dan misgaat. De gulden snede is de beroemde 'mooie' verdeling van een lijnstuk in twee stukken, zie Figuur 1.



$$\frac{a}{b} = \frac{a+b}{a} = \varphi \approx 1{,}618\ldots$$

FIGUUR 1. Bij een lijnstuk dat verdeeld is volgens de gulden snede verhoudt het grootste van de twee delen zich tot het kleinste, zoals het gehele lijnstuk zich verhoudt tot het grootste deel. Als we het langste stuk a noemen en het kortste b, dan hebben we $\varphi = \frac{a}{b} = \frac{a+b}{a}$. We vinden dat $a = b\varphi$ en invullen in $\varphi = \frac{a+b}{a}$ geeft $\varphi = \frac{b\,\varphi+b}{b\,\varphi} = \frac{\varphi+1}{\varphi}$. Dus $\varphi^2 - \varphi - 1 = 0$. Oplossen van deze vergelijking (bijvoorbeeld met de abc-formule) geeft als enige positieve oplossing $\varphi = \frac{1+\sqrt{5}}{2} \approx 1{,}61803399$.

We kunnen de kettingbreuk voor de gulden snede zonder het kettingbreukenrecept maken. Uit de vergelijking $\varphi^2 - \varphi - 1 = 0$ concluderen we dat $\varphi = 1 + \frac{1}{\varphi}$. Door aan de rechterkant herhaaldelijk $\varphi = 1 + \frac{1}{\varphi}$ in te vullen is de kettingbreuk van de gulden snede makkelijk te vinden:

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \frac{1}{1+\frac{1}{\varphi}} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\varphi}}}} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\dots}}}}}}.$$

We zien dat de kettingbreuk van de gulden snede uit alleen maar enen bestaat, daarom zijn de benaderingen de slechtst mogelijke.

Hurwitz bewees in 1891 (het jaar waarin Stanford University zijn deuren opende) de volgende stelling.

Stelling 3. *Voor elke irrationale x bestaan er oneindig veel breuken p/q die x met kwaliteit kleiner dan $\frac{1}{\sqrt{5}}$ benaderen:*

$$q^2 \left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}}.$$

De constante kan niet worden vervangen door een kleinere.

Merk op dat de stelling van Hurwitz direct volgt uit de resultaten van Legendre en Borel omdat $\frac{1}{\sqrt{5}} < \frac{1}{2}$.

8. Is dit hét kettingbreukalgoritme?

Hierboven maakten we de kettingbreuk voor x door steeds het gehele deel van $\frac{1}{x}$ te nemen. Maar we kunnen ook andere kettingbreuken maken. Bijvoorbeeld door $\frac{1}{x}$ af te ronden naar het dichtstbijzijnde gehele getal. Bij het benaderen van π kregen we in Stap 2) bijvoorbeeld $\frac{1}{0{,}06251330\dots} \approx 15{,}99659441$, toen namen we $a_2 = 15$. Zou het niet logischer zijn om af te ronden naar 16? Als we dat consequent doen, dan krijgen we voor π de volgende dichtstbijzijnde-gehele-getallen-kettingbreuk

$$\pi = 3 + \cfrac{1}{7+\cfrac{1}{16-\cfrac{1}{294+\dots}}}.$$

Afkappen geeft als eerste benaderingen $\frac{22}{7}$, $\frac{355}{113}$ en $\frac{104348}{33215}$. Deze kettingbreuk slaat de slechte benadering $\frac{333}{106}$ over, maar elke benadering die wordt gevonden, is er één die het reguliere kettingbreukalgoritme ook vindt. Het reguliere kettingbreukalgoritme vindt dus meer benaderingen.

Wie wiskundigen kent, weet dat zij het liefste alles, altijd en overal generaliseren. Dit leidt tot een op het eerste gezicht wat bizarre versie van het kettingbreukalgoritme: α-Rosen-kettingbreuken. In plaats van $\frac{1}{x}$ af te ronden, nemen we hierbij het gehele deel van $\left|\frac{1}{\lambda x}\right| + 1 - \alpha$, waarbij $\lambda = 2\cos\frac{\pi}{q}$ (voor een geheel getal $q \geq 3$) en waarbij α een reëel getal is tussen $\frac{1}{2}$ en $\frac{1}{\lambda}$. Deze kettingbreuken bestuderen we in Hoofdstuk III en IV van dit proefschrift. In 1954 (het jaar dat zowel *Lord of the flies* als *Lord of the rings* uitkwamen) introduceerde David Rosen de naar hem genoemde Rosen-kettingbreuken (met $\alpha = \frac{1}{2}$) om eigenschappen van bepaalde groepen te bestuderen. De stap naar α-Rosen-kettingbreuken is veel recenter: deze breuken werden slechts twee jaar geleden voor het eerst onderzocht door Dajani, Kraaikamp en Steiner.

Het grote voordeel van α-Rosen-kettingbreuken is dat ze allerlei andere soorten kettingbreuken omvatten. Als je een eigenschap van α-Rosen-kettingbreuken bewijst, dan heb je die eigenschap bijvoorbeeld ook onmiddellijk bewezen voor gewone kettingbreuken.

9. Hoe houd je alle informatie bij?

We kijken vanaf nu alleen naar irrationale getallen tussen 0 en 1. We benaderen immers toch alleen het niet-gehele deel van een getal met een kettingbreuk. Omdat het wat onhandig is om $\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}$ te schrijven, introduceren we de kortere notatie $[a_1, a_2, a_3, \ldots]$.

Als je het kettingbreukalgoritme steeds herhaalt, dan zagen we hierboven hoe je één voor één de getallen a_i krijgt. Maar verder raak je alle informatie over de rest van de kettingbreuk kwijt. Daarom introduceren we twee kettingbreuken: de *toekomst* t_n en het *verleden* v_n op plaats n.

$$t_n = [a_{n+1}, a_{n+2}, a_{n+3}, \ldots] \quad \text{and} \quad v_n = [a_n, a_{n-1}, a_{n-2}, \ldots, a_1].$$

De toekomst representeert alles wat er na a_n komt en omgekeerd representeert het verleden juist alles wat er tot en met a_n kwam.

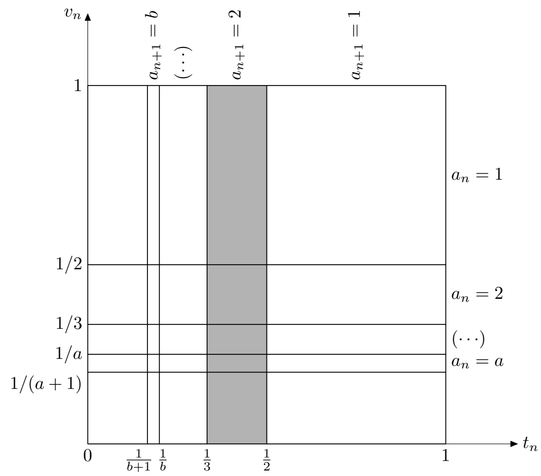
We gebruiken een twee-dimensionale functie $\mathcal{T}$ die (t_n, v_n) naar (t_{n+1}, v_{n+1}) stuurt. Zo blijft alle informatie bewaard: om van t_n naar t_{n+1} te gaan, hoef je alleen de eerste term a_{n+1} weg te gooien. Om van v_n naar v_{n+1} te gaan, moest je juist a_{n+1} aan het begin toevoegen.

In dit proefschrift maak ik veel gebruik van een meetkundige methode op het grootste gebied waarop $\mathcal{T}$ netjes werkt. We noemen dit de *natuurlijke uitbreiding*. We tekenen de natuurlijke uitbreiding in een assenstelsel met t_n op de x-as en v_n op y-as. Voor reguliere kettingbreuken is het een vierkant met zijde 1, zie figuur 2.

Voor de andere soorten kettingbreuken in dit proefschrift is de natuurlijke uitbreiding wat ingewikkelder, zie figuur 4 voor voorbeelden voor α-Rosen kettingbreuken.

10. Heb je ook kettingbreuken in hogere dimensies?

Als je wiskundigen wilt uitdagen, dan kun je altijd vragen of hun resultaten nog zijn uit te breiden, bijvoorbeeld naar hogere dimensies. Bij kettingbreuken is dit een pijnlijk punt. Er zijn namelijk wel allerlei generalisaties voor hogere dimensies, maar die hebben lang niet zulke mooie eigenschappen als het reguliere kettingbreukalgoritme.



FIGUUR 2. De natuurlijke uitbreiding voor reguliere kettingbreuken. Op de getekende horizontale strips is a_n constant, op elke verticale strip is a_{n+1} constant. In de grijze strip geldt bijvoorbeeld dat $t_n = \frac{1}{a_{n+1}+\ldots}$ groter is dan $\frac{1}{3}$ en kleiner dan $\frac{1}{2}$. Dus deze strip bevat alle punten (t_n, v_n) waarvoor $a_{n+1} = 2$.

Bij reguliere kettingbreuken zoeken we voor een willekeurige x een breuk $\frac{p}{q}$ die dicht bij x ligt. In hogere dimensies kunnen we de volgende twee kanten op.

De eerste optie is om niet één maar meer getallen te benaderen met breuken met dezelfde noemer. Je hebt dan een aantal (zeg m) irrationale getallen $x_1, x_2, \ldots, x_m$ en zoekt $m+1$ gehele getallen $p_1, p_2, \ldots, p_m$ en q zodat de breuk $\frac{p_1}{q}$ dicht bij x_1 ligt, $\frac{p_2}{q}$ dicht bij x_2, enzovoorts.

De tweede optie is om voor een aantal (zeg n) getallen $x_1, x_2, \ldots, x_n$ in totaal $n+1$ gehele getallen $p, q_1, \ldots, q_n$ te zoeken zodat $q_1x_1 + q_2x_2 + \cdots + q_nx_n$ dicht bij p ligt.

Je kunt de twee opties combineren door voor $m \times n$ gegeven getallen $x_{11}, \ldots, x_{mn}$ te zoeken naar $m + n$ gehele getallen $p_1, p_2, \ldots, p_m$ en $q_1, \ldots, q_n$ zodat de som $q_1x_{i1} + q_2x_{i2} + \cdots + q_nx_{in}$ dicht bij p_i ligt voor elke i tussen 1 en m. Meerdimensionale benaderingen hebben toepassingen van jpeg-compressie tot het oplossen van optimaliseringsproblemen.

Nu nemen we voor de kwaliteit het grootste verschil tussen één van de i sommen en de bijbehorende p_i, vermenigvuldigd met de grootste van de q_j's tot de macht m/n. Of zoals wiskundigen het schrijven:

$$q^{\frac{m}{n}} \max_i |q_1\, x_{i1} + \cdots + q_m\, x_{im} - p_i| \qquad \text{waarbij} \quad q = \max_j |q_j|.$$

Een benadering heet weer goed als de kwaliteit klein is. Als $m = n = 1$, dan is dit precies de kwaliteit van een benadering die we eerder gebruikten.

In 1842 (het jaar dat *Nabucco* van Verdi in première ging) bewees Dirichlet het volgende.

Stelling 4. *Voor elke gegeven* $x_{11}, \ldots, x_{mn}$ *bestaan er oneindig veel benaderingen met kwaliteit kleiner dan 1.*

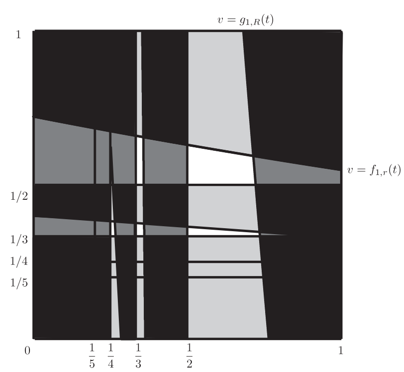
Voor de wiskundigen die dit toch stiekem lezen: we nemen aan dan er minstens één i *is waarvoor* $1, x_{i1}, \ldots, x_{im}$ *lineair onafhankelijk zijn over* $\mathbb{Q}$.

Dirichlet bewees deze stelling met zijn beroemde ladenprincipe: als je meer dan n balletjes verdeelt over n laden, dan is er minstens één lade met meer dan één balletje erin. Zijn bewijs geeft helaas geen goede methode om benaderingen met kwaliteit kleiner dan 1 te vinden. Ruim 150 jaar later hebben we nog steeds geen efficiënt algoritme gevonden om alle benaderingen met kwaliteit kleiner dan 1 te geven. Behalve als $m = n = 1$ natuurlijk, want dan kunnen we het kettingbreukalgoritme gebruiken.

11. Wat staat er in dit proefschrift?

In het inleidende hoofdstuk I geef ik definities van de gebruikte begrippen, klassieke stellingen en de belangrijkste resultaten uit dit proefschrift

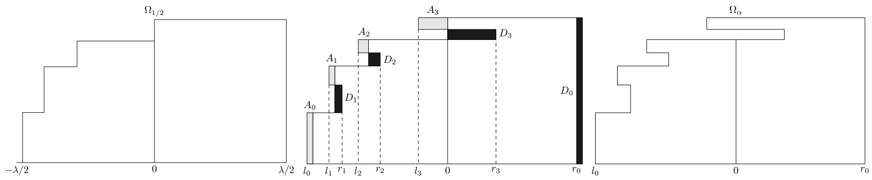
In hoofdstuk II kijk ik naar reguliere kettingbreuken, maar gebruik ik een andere kwaliteitsmaat voor wat een goede benadering is. De vraag die ik voor deze maat beantwoord is: stel dat $n-1$-ste en $n+1$-ste benaderingen heel goed zijn, wat kun je dan zeggen over de kwaliteit van de n-de benadering die daar tussenin zit? En andersom: hoe goed moet een benadering zijn die tussen twee slechte benaderingen inzit? Daarnaast bereken ik ook de kans dat zulke situaties voorkomen.



Figuur 3. Een voorbeeld van de meetkundige methode uit hoofdstuk II, zie voor meer uitleg bladzijde 27.

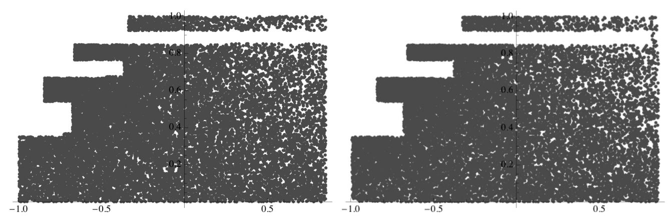
In Hoofdstuk III kijk ik zoals gezegd naar α-Rosen-kettingbreuken. Ik generaliseer de stellingen van Borel (die het minimum van de kwaliteit in een reeks opeenvolgende benaderingen geeft) en Hurwitz (die de kleinste kwaliteit geeft die oneindig vaak voorkomt) voor deze kettingbreuken.

Ook Hoofdstuk IV gaat over α-Rosen-kettingbreuken. Ik bepaal de kleinste waarde voor α waarbij de natuurlijke uibreiding nog één gebied vormt – als α te klein wordt,



FIGUUR 4. Een voorbeeld van *quilten*. Links staat de natuurlijke uitbreiding voor α-Rosen-kettingbreuken met $\alpha = \frac{1}{2}$. In het midden hebben we aangegeven welke rechthoeken weg moeten (zwart) en welke rechthoeken erbij moeten (grijs) om de natuurlijke uitbreiding voor een α-Rosen-kettingbreuk met zekere $\alpha < \frac{1}{2}$ te vinden. Het resultaat staat rechts.

dan valt het gebied in stukken uit elkaar; zie figuur 5. We vinden de natuurlijke uitbreiding met een techniek die we *quilten* noemen: we beginnen met de al bekende natuurlijke uitbreiding voor het geval $\alpha = \frac{1}{2}$ en plakken daar rechthoeken aan en halen daar rechthoeken af. We kunnen door deze constructie ook eigenschappen van de natuurlijke uitbreiding voor een α-Rosen-kettingbreuk afleiden. Zodra de natuurlijke uitbreiding in twee stukken uit elkaar valt, verandert bijvoorbeeld de entropie van het systeem.



FIGUUR 5. Simulaties van de natuurlijke uitbreiding van α-Rosen-kettingbreuken. Links is α te klein, waardoor de natuurlijke uitbreiding in twee losse stukken uit elkaar valt. Rechts is α iets groter en zit er een verbinding tussen de twee delen die in het linkerplaatje los zijn.

In Hoofdstuk V geef ik een meerdimensionaal kettingbreukalgoritme dat efficiënt benaderingen zoekt met een gegarandeerde kwaliteit die alleen afhangt van de dimensies m en n. Uit de benaderingen die dit algoritme vindt, leid ik een ondergrens af voor de kwaliteit van alle mogelijke benaderingen tot een bepaalde grens voor q. Tenslotte toon ik experimentele data voor de verdeling van de kwaliteiten in meer dimensies.

Dankwoord

I color the sky with you,
I let you choose the blue.
Everyone needs an editor - Mates of State

Allereerst mijn dank aan alle wiskundigen op het Mathematisch Instituut en daarbuiten voor alle inspiratie, de al dan niet felle discussies en de liefde voor ons vak.

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Beste Cor, al in mijn eerste collegejaar in Delft was ik fan van je. Door jouw enthousiasme raakte ik in de ban van kettingbreuken en de mooiste momenten van mijn onderzoek waren die met jou voor een schoolbord. Daarnaast ben je inmiddels een heel goede vriend geworden en ik hoop dat we nog heel lang samen leuke dingen zullen doen — al dan niet met kettingbreuken.

Curriculum vitae

Op 8 oktober 1979 werd Ionica Smeets geboren in Delft. Ze haalde in 1998 haar VWO-diploma op het Maascollege in Maassluis. Na lang twijfelen tussen alfa en bèta koos ze voor Technische Informatica in Delft. In 1999 haalde ze haar propedeuse en stapte prompt over naar Technische Wiskunde. Ze vond de wiskundevakken namelijk significant leuker dan alle andere. Cor Kraaikamp maakte haar enthousiast voor kettingbreuken en onder zijn begeleiding schreef ze haar afstudeerscriptie *Metric and arithmetic properties of Rosen continued fractions.* Tijdens haar afstuderen was ze twee maanden te gast bij Thomas A. Schmidt aan Oregon State University. Ze studeerde in januari 2005 cum laude af.

Tijdens haar studie schreef ze voor universiteitskrant Delta, werkte ze bij een bibliotheek en was ze pr-medewerker van Theater Schuurkerk in Maassluis. Daarnaast volgde ze met wisselend succes cursussen kleinkunst, fotografie en stemvorming.

Van december 2004 tot mei 2007 was Ionica Smeets de eerste *Nationale PR-medewerker wiskunde.* In die functie zette ze zich één dag per week in voor de verbetering van het imago van wiskunde en schreef ze artikelen voor Kennislink. Tussen januari en september 2005 maakte ze voor de Universiteit van Amsterdam een onderzoeksrapport over de rol van wiskundigen in het bedrijfsleven.

In september 2005 begon ze als Assistent in Opleiding op het Mathematisch Instituut aan de Universiteit Leiden onder begeleiding van Robert Tijdeman en Cor Kraaikamp. Naast haar onderzoek schreef Smeets als wetenschapsjournalist voor onder andere NRC Handelsblad en Natuurwetenschap & Techniek. Sinds januari 2008 schrijft ze een maandelijkse column voor Technisch Weekblad.

Samen met Jeanine Daems maakte ze furore als de wiskundemeisjes. Hun gelijknamige weblog trekt ruim 2500 bezoekers per dag en werd meermaals bekroond, onder andere met twee Dutch Bloggies en de Mr. K.J. Cathprijs. De wiskundemeisjes gaven voordrachten van Lowlands tot Kortrijk. Ze hadden een eigen column bij schooltelevisie en presenteerden in 2007 met Joost Prinsen de Nationale Rekentoets. Sinds januari 2009 hebben de wiskundemeisjes een tweewekelijkse column in De Volkskrant.

Sinds 1 oktober 2009 doet Smeets, als part-time post-doc aan de Universiteit Leiden, met Bas Haring onderzoek naar publiek begrip van wetenschap. De rest van de week schrijft ze artikelen, geeft ze voordrachten en presenteert ze wetenschapsdagen.